var-1

# VARIANCES

Gamma Distribution

→ Chi-Square Distribution

Beta Distribution

→ F Distribution

Confidence Interval

Inference - One Variance

Inference - Two Variances

# GAMMA DISTRIBUTION

**Used to describe random variables bounded at one end**

**Most appropriate model for the time required for a total of exactly η independent events to take place if events occur at a constant rate λ (e.g., the time between consecutive maintenance operations for an aircraft that is inspected after every η missions)**

**Many phenomena that cannot be justified theoretically as gamma variates have been found empirically to be well approximated by the model (e.g., the distribution of family income)**

**Probability Density Function:**

$$f(x;\eta,\lambda) = \begin{cases} \dfrac{\lambda^{\eta}}{\Gamma(\eta)} x^{\eta-1} e^{-\lambda x} \\ 0 \text{ elsewhere} \end{cases}$$

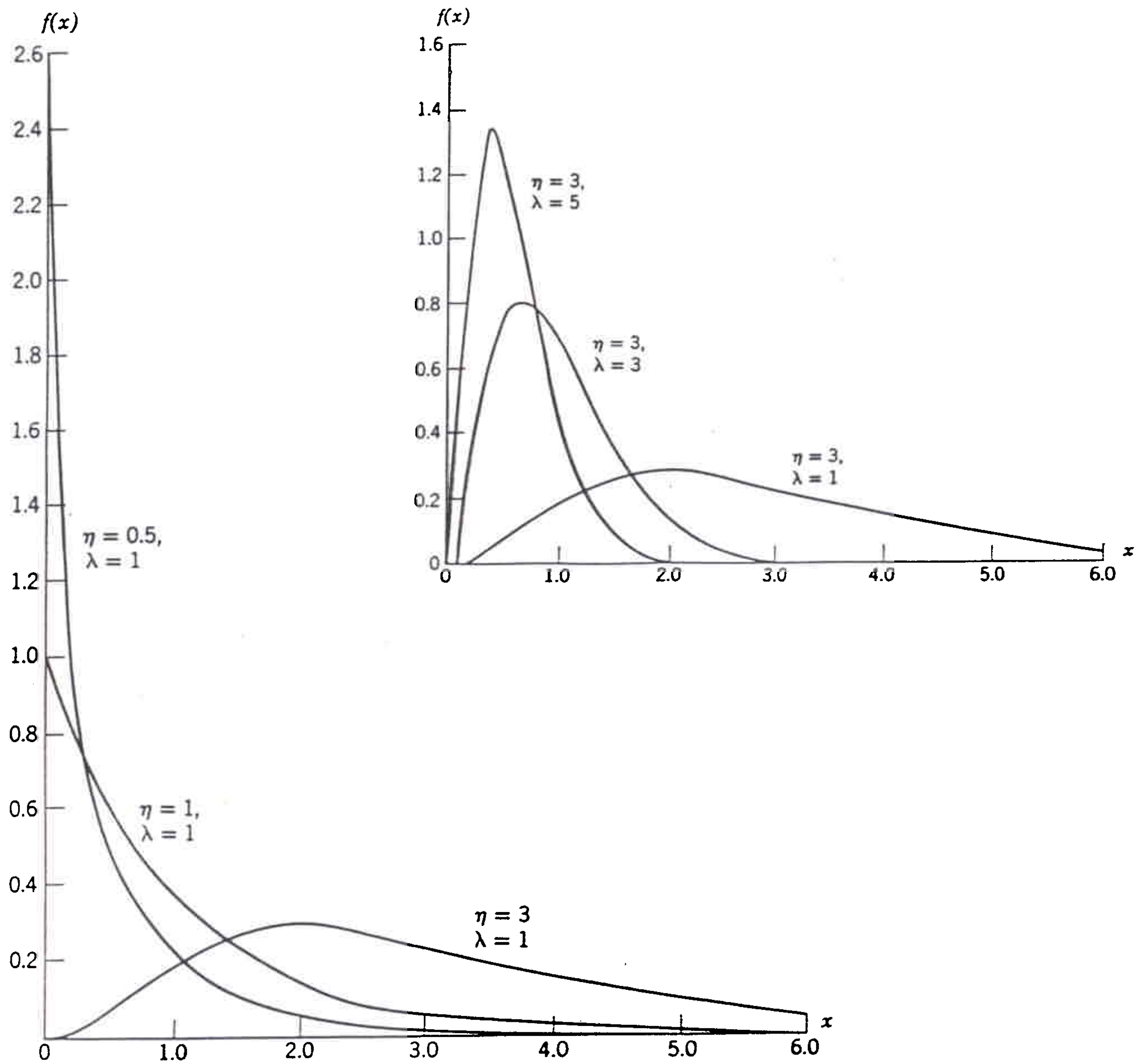
$$x \geq 0; \quad \lambda > 0; \quad \eta > 0$$

where $\Gamma(\eta) = \Gamma(\eta) \int_0^{\infty} x^{\eta-1} e^{-x}\,dx$

and $\Gamma(\eta) = (\eta-1)!$ when $\eta$ is a positive integer

var-3

# GAMMA DISTRIBUTION



# CHI-SQUARE DISTRIBUTION

## ($\chi^2$ Distribution)

Special case of the Gamma Distribution when $\lambda = ½$ and $\eta$ is a multiple of ½

The single parameter is the integer, $\gamma = 2\eta$, generally referred to as its degrees of freedom

Applications arise from the fact that the sum of the squares of the values of $\gamma$ random observations from a standardized Normal Distribution has a Chi-Square Distribution with $\gamma$ degrees of freedom

Probability density:

$$f(x;\gamma) = \begin{cases} \dfrac{(1/2)^{\gamma/2}}{\Gamma(\gamma/2)} x^{\gamma/2 - 1} e^{-x/2} \\ 0 \text{ elsewhere} \end{cases}$$

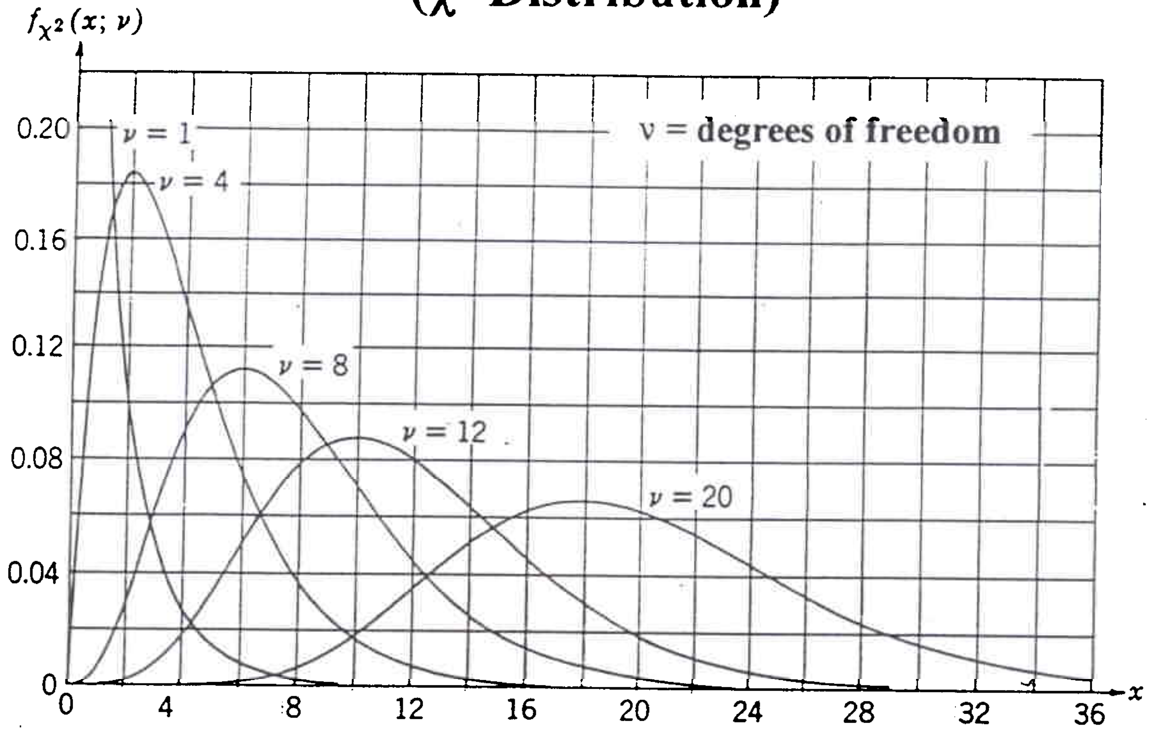
$x \geq 0$ ; $\gamma$ a positive integer

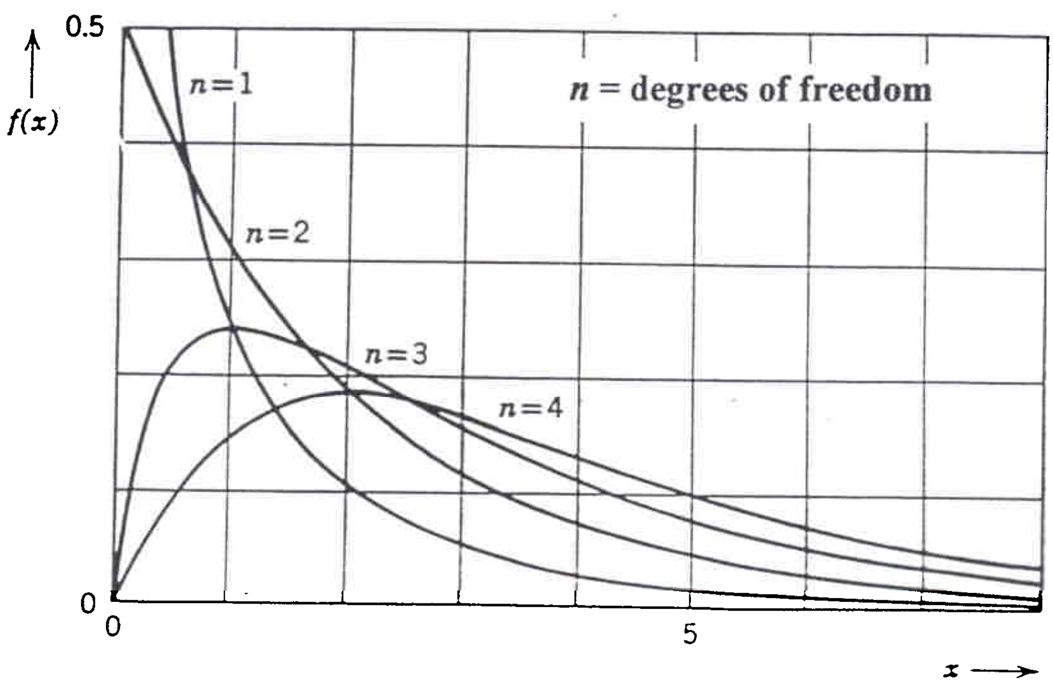
Mean = $\gamma$ and Variance = $2\gamma$

var-5

# CHI-SQUARE DISTRIBUTION

## ($\chi^2$ Distribution)



var-6

# CHI-SQUARE DISTRIBUTION

## ($\chi^2$ Distribution)

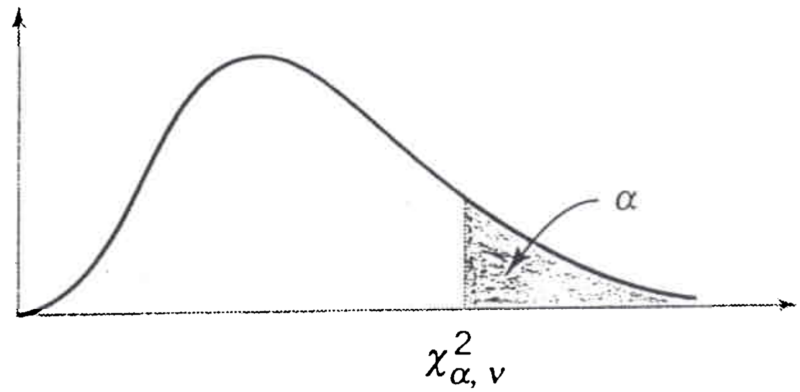



Table III Percentage Points $\chi^2_{\alpha,\nu}$ of the Chi-Squared Distribution

| $\nu$ \ $\alpha$ | .995 | .990 | .975 | .950 | .900 | .500 | .100 | .050 |
|---|---|---|---|---|---|---|---|---|
| 1 | .00+ | .00+ | .00+ | .00+ | .02 | .45 | 2.71 | 3.8 |
| 2 | .01 | .02 | .05 | .10 | .21 | 1.39 | 4.61 | |
| 3 | .07 | .11 | .22 | .35 | .58 | 2.37 | 6. | |
| 4 | .21 | .30 | .48 | .71 | 1.06 | 3.36 | | |
| 5 | .41 | .55 | .83 | 1.15 | 1.61 | 4.35 | | |
| 6 | .68 | .87 | 1.24 | 1.64 | 2.20 | 5.35 | | |
| 7 | .99 | 1.24 | 1.69 | 2.17 | 2.83 | 6. | | |
| 8 | 1.34 | 1.65 | 2.18 | 2.73 | 3.49 | | | |
| 9 | 1.73 | 2.09 | 2.70 | 3.33 | | | | |
| 10 | 2.16 | 2.56 | | | | | | |
| 11 | 2.60 | | | | | | | |
| 12 | [illegible] | | | | | | | |

var-7

# CHI-SQUARE DISTRIBUTION

## ($\chi^2$ Distribution)

*Table* 5 *Values of* $\chi^2_\alpha$

| $\nu$ | $\alpha = 0.995$ | $\alpha = 0.99$ | $\alpha = 0.975$ | $\alpha = 0.95$ |
|---|---|---|---|---|
| 1 | 0.0000393 | 0.000157 | 0.000982 | 0.0039[illegible] |
| 2 | 0.0100 | 0.0201 | 0.0506 | 0.10[illegible] |
| 3 | 0.0717 | 0.115 | 0.216 | [illegible] |
| 4 | 0.207 | 0.297 | 0.484 | |
| 5 | 0.412 | 0.554 | 0.831 | |
| 6 | 0.676 | 0.872 | 1.237 | |
| 7 | 0.989 | 1.239 | 1.690 | |
| 8 | 1.344 | 1.646 | 2.180 | |
| 9 | 1.735 | 2.088 | 2.700 | |
| 10 | 2.156 | 2.558 | 3.2[illegible] | |
| 11 | 2.603 | 3.053 | | |
| 12 | 3.074 | 3.571 | | |
| 13 | 3.[illegible] | [illegible] | | |

# BETA DISTRIBUTION

Useful model for variates whose values are limited to a finite interval

Used to represent a wide variety of physical variables such as the daily proportion of defective units on a production line and estimated time to complete a project phase (in PERT)

Probability Density Function defined over the interval (0,1):

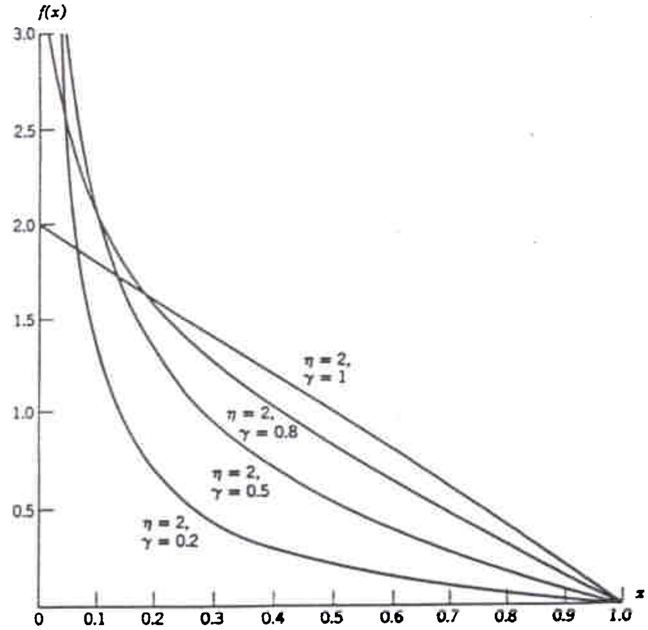
$$f(x;\gamma,\eta)=\begin{cases}\dfrac{\Gamma(\gamma+\eta)}{\Gamma(\gamma)\Gamma(\eta)}\,x^{\gamma-1}(1-x)^{\eta-1}\\ 0 \text{ elsewhere}\end{cases}$$

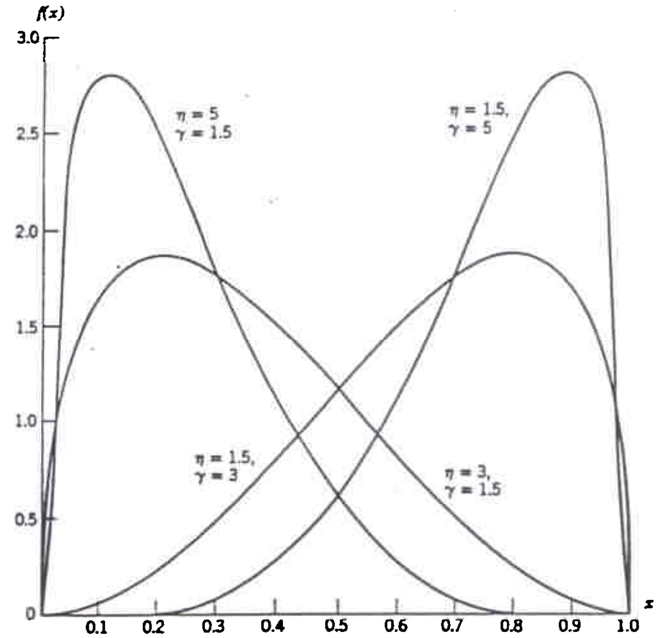
$$0\leq x\leq 1;\quad 0<\gamma;\quad 0<\eta$$

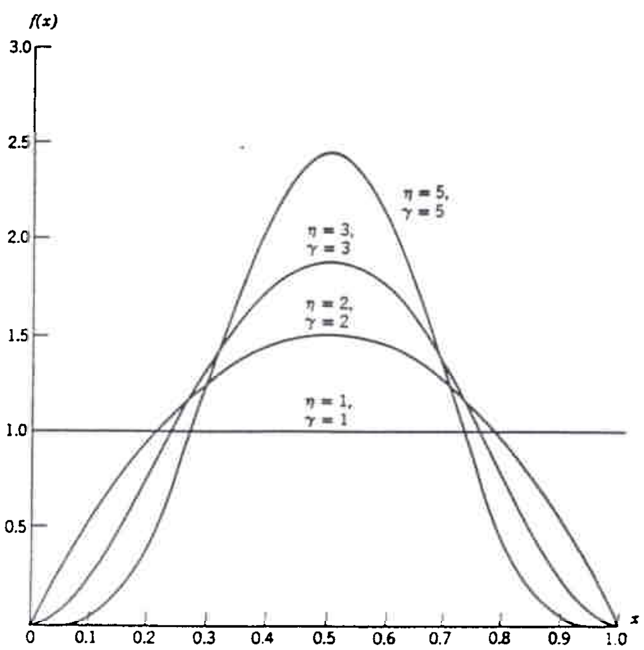
**Mean** = $\gamma/(\eta+\gamma)$
**Variance** = $\eta\gamma / \{ (\eta+\gamma)^2 (\eta+\gamma+1) \}$

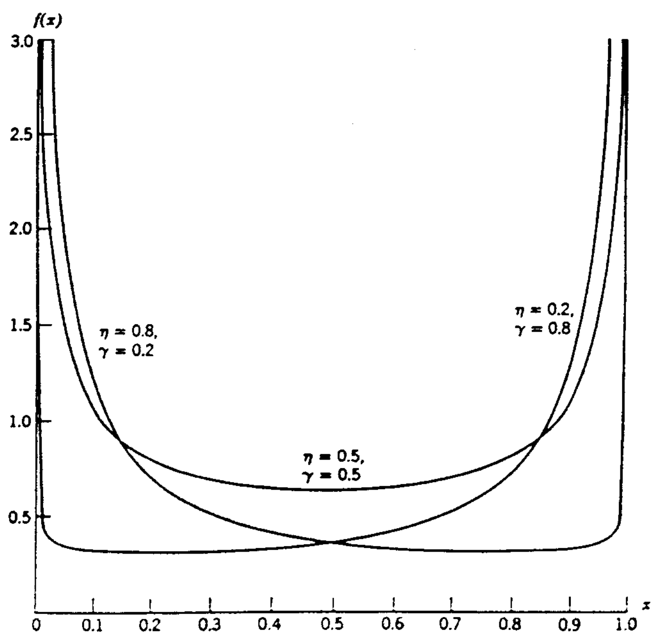
var-9

# BETA DISTRIBUTION



# F-DISTRIBUTION

**Primary use is as a sampling model of Normal statistics**

**Formed from the ratio of two Chi-Square Distributions which, with an appropriate transformation, is a form of the Beta Distribution**

**Use is highly dependent on confirmation of population Normality of original sampled variates**

**Probability Density Function, with *(m, n)* degrees of freedom is given by**

$$f(y; m, n) = \frac{\Gamma\{(m+n)/2\}}{\Gamma(m/2)\Gamma(n/2)} \cdot \left\{\frac{m}{n}\right\}^{m/n} \cdot \frac{y^{(m/2)-1}}{\{1 + (m/n)y\}^{(m+n)/2}}$$

$$0 < y; \quad 0 < m, n$$

$$\text{Mean} = n/(n-2), \quad n > 2$$

$$\text{Variance} = \frac{2n^2(m+n-2)}{m(n-2)^2(n-4)}, \quad n > 4$$

# F - DISTRIBUTION

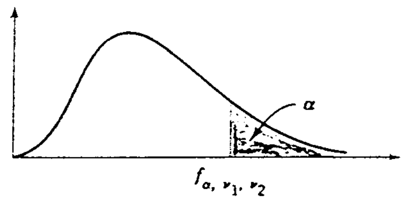



Table V Percentage Points $f_{\alpha,\nu_1,\nu_2}$ of the $F$-Distribution

$f_{0.25,\nu_1,\nu_2}$

| $\nu_2$ \ $\nu_1$ | Degrees of freedom for the numerator ($\nu_1$) | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 12 | 15 | 20 | 24 | 30 |
| 1 | 5.83 | 7.50 | 8.20 | 8.58 | 8.82 | 8.98 | 9.10 | 9.19 | 9.26 | 9.32 | 9.41 | 9.49 | 9.58 | 9.63 | 9.67 |
| 2 | 2.57 | 3.00 | 3.15 | 3.23 | 3.28 | 3.31 | 3.34 | 3.35 | 3.37 | 3.38 | 3.39 | 3.41 | 3.43 | 3.43 | 3.4[illegible] |
| 3 | 2.02 | 2.28 | 2.36 | 2.39 | 2.41 | 2.42 | 2.43 | 2.44 | 2.44 | 2.44 | 2.45 | 2.46 | 2.46 | 2.46 | 2.4[illegible] |
| 4 | 1.81 | 2.00 | 2.05 | 2.06 | 2.07 | 2.08 | 2.08 | 2.08 | 2.08 | 2.08 | 2.08 | 2.08 | 2.08 | 2.08 | 2.[illegible] |
| 5 | 1.69 | 1.85 | 1.88 | 1.89 | 1.89 | 1.89 | 1.89 | 1.89 | 1.89 | 1.89 | 1.89 | 1.89 | 1.88 | 1.88 | [illegible] |
| 6 | 1.62 | 1.76 | 1.78 | 1.79 | 1.79 | 1.78 | 1.78 | 1.78 | 1.77 | 1.77 | 1.77 | 1.76 | 1.76 | 1.75 | |
| 7 | 1.57 | 1.70 | 1.72 | 1.72 | 1.71 | 1.71 | 1.70 | 1.70 | 1.70 | 1.69 | 1.68 | 1.68 | 1.67 | 1.67 | |
| 8 | 1.54 | 1.66 | 1.67 | 1.66 | 1.66 | 1.65 | 1.64 | 1.64 | 1.63 | 1.63 | 1.62 | 1.62 | 1.61 | [illegible] | |
| 9 | 1.51 | 1.62 | 1.63 | 1.63 | 1.62 | 1.61 | 1.60 | 1.60 | 1.59 | 1.59 | 1.58 | 1.57 | 1.56 | | |
| 10 | 1.49 | 1.60 | 1.60 | 1.59 | 1.59 | 1.58 | 1.57 | 1.56 | 1.56 | 1.55 | 1.54 | 1.53 | | | |
| 11 | 1.47 | 1.58 | 1.58 | 1.57 | 1.56 | 1.55 | 1.54 | 1.53 | 1.53 | 1.52 | 1.51 | 1.50 | | | |
| 12 | 1.46 | 1.56 | 1.56 | 1.55 | 1.54 | 1.53 | 1.52 | 1.51 | 1.51 | 1.50 | 1.49 | 1.48 | | | |
| 13 | 1.45 | 1.55 | 1.55 | 1.53 | 1.52 | 1.51 | 1.50 | 1.49 | 1.49 | 1.48 | 1.47 | 1.46 | | | |
| 14 | 1.44 | 1.53 | 1.53 | 1.52 | 1.51 | 1.50 | 1.49 | 1.48 | 1.47 | 1.46 | 1.45 | 1.44 | | | |
| 15 | 1.43 | 1.52 | 1.52 | 1.51 | 1.49 | 1.48 | 1.47 | 1.46 | 1.46 | 1.45 | 1.44 | 1.[illegible] | | | |
| 16 | 1.42 | 1.51 | 1.51 | 1.50 | 1.48 | 1.47 | 1.46 | 1.45 | 1.44 | 1.44 | 1.43 | | | | |
| 17 | 1.42 | 1.51 | 1.50 | 1.49 | 1.47 | 1.46 | 1.45 | 1.44 | 1.43 | 1.43 | [illegible] | | | | |
| 18 | 1.41 | 1.50 | 1.49 | 1.48 | 1.46 | 1.45 | 1.44 | [illegible] | | | | | | | |
| 19 | 1.41 | 1.49 | 1.49 | 1.47 | 1.46 | 1.44 | 1.43 | | | | | | | | |
| 20 | 1.40 | 1.49 | 1.48 | 1.47 | 1.45 | 1.44 | 1.[illegible] | | | | | | | | |
| 21 | 1.40 | 1.48 | 1.48 | 1.46 | 1.44 | 1.43 | | | | | | | | | |
| 22 | 1.40 | 1.48 | 1.47 | 1.45 | 1.44 | 1.42 | | | | | | | | | |
| 23 | 1.39 | 1.47 | 1.47 | 1.45 | 1.43 | 1.42 | | | | | | | | | |
| 24 | 1.39 | 1.47 | 1.46 | 1.44 | 1.43 | | | | | | | | | | |
| 25 | 1.39 | 1.47 | 1.46 | [illegible] | | | | | | | | | | | |
| 26 | 1.38 | [illegible] | | | | | | | | | | | | | |

Degrees of freedom for the denominator ($\nu_2$)

var-12

# F-DISTRIBUTION

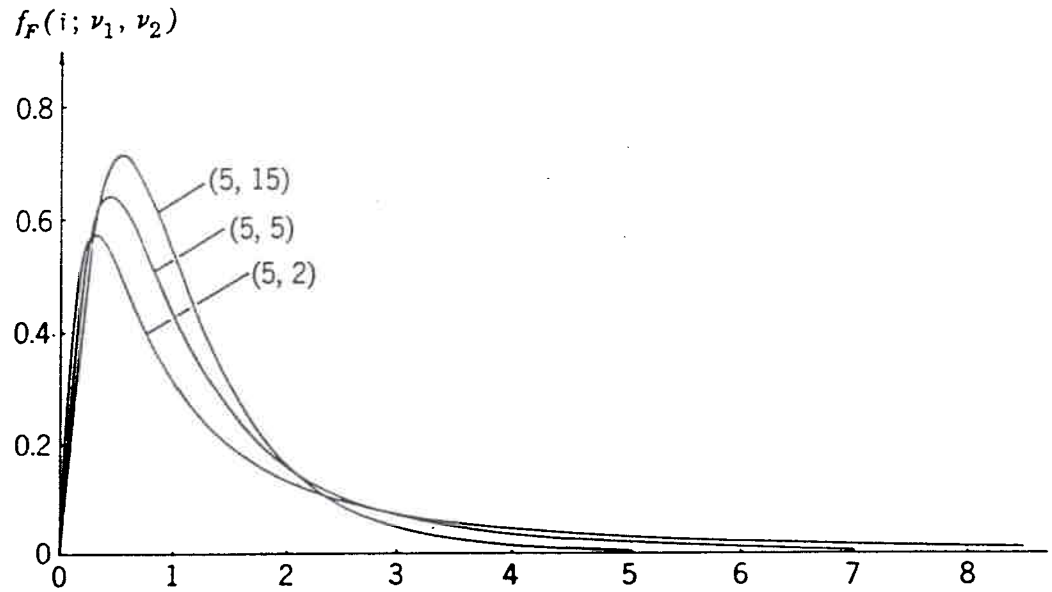



Table 6(a) Values of $F_{0.05}$

| $\nu_2$ = Degrees of freedom for denominator | $\nu_1$ = Degrees of freedom for numerator | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 12 | 15 | 20 |
| 1 | 161 | 200 | 216 | 225 | 230 | 234 | 237 | 239 | 241 | 242 | 244 | 246 | [illegible] |
| 2 | 18.50 | 19.00 | 19.20 | 19.20 | 19.30 | 19.30 | 19.40 | 19.40 | 19.40 | 19.40 | 19.40 | 19.40 | [illegible] |
| 3 | 10.10 | 9.55 | 9.28 | 9.12 | 9.01 | 8.94 | 8.89 | 8.85 | 8.81 | 8.79 | 8.74 | 8.70 | [illegible] |
| 4 | 7.71 | 6.94 | 6.59 | 6.39 | 6.26 | 6.16 | 6.09 | 6.04 | 6.00 | 5.96 | 5.91 | 5.86 | [illegible] |
| 5 | 6.61 | 5.79 | 5.41 | 5.19 | 5.05 | 4.95 | 4.88 | 4.82 | 4.77 | 4.74 | 4.68 | 4.62 | [illegible] |
| 6 | 5.99 | 5.14 | 4.76 | 4.53 | 4.39 | 4.28 | 4.21 | 4.15 | 4.10 | 4.06 | 4.00 | [illegible] | [illegible] |
| 7 | 5.59 | 4.74 | 4.35 | 4.12 | 3.97 | 3.87 | 3.79 | 3.73 | 3.68 | 3.64 | [illegible] | [illegible] | [illegible] |
| 8 | 5.32 | 4.46 | 4.07 | 3.84 | 3.69 | 3.58 | 3.50 | 3.44 | 3.39 | [illegible] | [illegible] | [illegible] | [illegible] |
| 9 | 5.12 | 4.26 | 3.86 | 3.63 | 3.48 | 3.37 | 3.29 | 3.23 | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] |
| 10 | 4.96 | 4.10 | 3.71 | 3.48 | 3.33 | 3.22 | 3.14 | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] |
| 11 | [illegible] | [illegible] | 3.59 | 3.36 | 3.20 | 3.09 | 3.01 | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] |
| 12 | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | 3.00 | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] |
| 13 | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] | [illegible] |

# VARIANCES

*The sample variance*

$$S^2 = \frac{\sum_{i=1}^{n} (X_i - \overline{X})^2}{n - 1}$$

*is an unbiased estimator of $\sigma^2$*

***The mean of the sampling distribution of $s^2$ is given by $\sigma^2$***

# CONFIDENCE INTERVAL for VARIANCE of a NORMAL POPULATION

If $s^2$ is the variance of a sample of size $n$ from the distribution $N\sim(\mu, \sigma^2)$, then $(n-1)s^2/\sigma^2$ has the $\chi^2$ distribution with $(n-1)$ degrees of freedom.

Thus we can make the statement

$$P\left[\chi^2_{(n-1),(1-\alpha/2)} \le \frac{(n-1)s^2}{\sigma^2} \le \chi^2_{(n-1),(\alpha/2)}\right] = (1-\alpha)$$

Therefore the $100(1-\alpha)\%$ Confidence Interval for $\sigma^2$ is

$$\frac{(n-1)s^2}{\chi^2_{(n-1),(\alpha/2)}} \le \sigma^2 \le \frac{(n-1)s^2}{\chi^2_{(n-1),(1-\alpha/2)}}$$

Also, for $\sigma$:

$$\text{CONF}_{100(1-\alpha)\%}\left\{\sqrt{\frac{(n-1)s^2}{\chi^2_{,n-1,\alpha/2}}} \le \sigma \le \sqrt{\frac{(n-1)s^2}{\chi^2_{,n-1,1-\alpha/2}}}\right\}$$

# EXAMPLE

**IQ scores are usually accepted to be normally distributed with a variance of 225. Nine students in a certain high school were randomly selected and tested. Their IQ scores were:**

**93, 95, 98, 100, 105, 109, 110, 123, and 130.**

**Determine the 95 percent confidence interval around the population variance.**

$n = 9$ ; $s^2 = 159$

95% Confidence Interval

$\Rightarrow \alpha = 0.05 \Rightarrow \alpha/2 = 0.025$

Need $\chi^2$ values corresponding to 0.975 and 0.025 with $(n-1) = 8$ degrees of freedom

Critical values — from $\chi^2$ Table: $c_1 = 2.180$ (for $1-\alpha/2$) $\chi^2_{8,\,0.975}$

and $c_2 = 17.535$ (for $\alpha/2$) $\chi^2_{8,\,0.025}$

$k_1 = (n-1)s^2/c_1 = (8)(159)/2.180 = 583.5$

$k_2 = (n-1)s^2/c_2 = (8)(159)/17.535 = 72.5$

Thus $\text{CONF}_{95\%}\left\{ \underset{(k_2)}{72.5} \le \sigma^2 \le \underset{(k_1)}{583.5} \right\}$

$$\frac{(n-1)s^2}{\chi^2_{dof,\ \alpha/2 \text{ or } 1-\alpha/2}}$$

$\sqrt{72.5} \le \sigma \le \sqrt{583.5}$

# HYPOTHESIS TEST FOR VARIANCE OF A NORMAL POPULATION

Test Statistic: $\chi^2 = \dfrac{(n-1)s^2}{\sigma_0^2}$

CASE 1: $H_0: \sigma^2 = \sigma_0^2$
$H_1: \sigma^2 \neq \sigma_0^2$

$c_1 = \chi^2$ value for $(1-\alpha/2)$ with $(n-1)$ dof
$c_2 = \chi^2$ value for $(\alpha/2)$ with $(n-1)$ dof

Reject Ho if $\chi^2_{test} < c_1$ or $\chi^2_{test} > c_2$

CASE 2: $H_0: \sigma^2 = \sigma_0^2$
$H_1: \sigma^2 < \sigma_0^2$

$c = \chi^2$ value for $(1-\alpha)$ with $(n-1)$ dof

Reject Ho if $\chi^2_{test} < c$

CASE 3: $H_0: \sigma^2 = \sigma_0^2$
$H_1: \sigma^2 > \sigma_0^2$

$c = \chi^2$ value for $(\alpha)$ with $(n-1)$ dof

Reject Ho if $\chi^2_{test} > c$

# HYPOTHESIS TEST FOR VARIANCE OF A NORMAL POPULATION
## *EXAMPLE*

The lapping process which is used to grind certain silicon wafers to the proper thickness is acceptable only if the population standard deviation is at most 0.50 mil. A sample of 15 wafers is selected, and the standard deviation of the sample is calculated to be 0.64 mil. If $\alpha = 0.05$, determine if the lapping process is unsatisfactory.

$H_0: \sigma^2 = \sigma_0^2$ $\qquad \sigma_0 = 0.50 \qquad \alpha = 0.05$

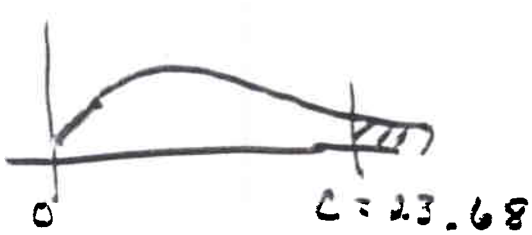
$H_1: \sigma^2 > \sigma_0^2$ $\qquad s = 0.64 \qquad n = 15$

$$\chi^2_{test} = \frac{(n-1)s^2}{\sigma_0^2} = \frac{(14)(0.64)^2}{(0.50)^2} = 22.94$$

From Table: At $\alpha = 0.05$, with $(n-1) = 14$ dof, $\chi^2_{crit} = 23.68$

Since $\chi^2_{test} < \chi^2_{crit}$,

We cannot reject $H_0$.



# HYPOTHESIS TEST FOR VARIANCES OF TWO POPULATIONS

Test Statistic: $\mathcal{F} = \dfrac{S_1^2}{S_2^2}$ numerator dof / denominator dof

CASE 1: $H_0: \sigma_1^2 - \sigma_2^2 = d$ $\quad H_0: \sigma_1^2 = \sigma_2^2$ when $d = 0$

$H_1: \sigma_1^2 - \sigma_2^2 \neq d$

$c_1 = \mathcal{F}$ value for $(1 - \alpha/2)$ with $(n_1 - 1, n_2 - 1)$ dof

$c_2 = \mathcal{F}$ value for $(\alpha/2)$ with $(n_1 - 1, n_2 - 1)$ dof

Reject Ho if $\mathcal{F}_{test} < c_1$ or $\mathcal{F}_{test} > c_2$

CASE 2: $H_0: \sigma_1^2 - \sigma_2^2 = d$

$H_1: \sigma_1^2 - \sigma_2^2 < d$

$c = \mathcal{F}$ value for $(1 - \alpha)$ with $(n_1 - 1, n_2 - 1)$ dof

Reject Ho if $\mathcal{F}_{test} < c$

Reject $H_0 \Rightarrow H_1$ is probably true

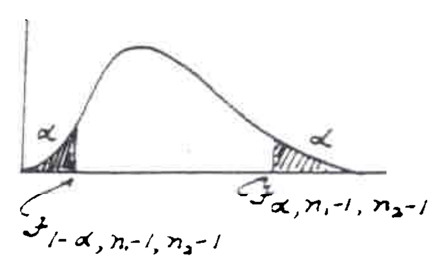
Do Not Reject $H_0 \Rightarrow$ Ho is probably true.

CASE 3: $H_0: \sigma_1^2 - \sigma_2^2 = d$

$H_1: \sigma_1^2 - \sigma_2^2 > d$

$c = \mathcal{F}$ value for $(\alpha)$ with $(n_1 - 1, n_2 - 1)$ dof

Reject Ho if $\mathcal{F}_{test} > c$



# F - DISTRIBUTION
## obtaining values for two tails

**Note that, in the F-Distribution tables, the values of F that are provided are for UPPER-TAIL probabilities ONLY.**

**In order to obtain F-Distribution values for LOWER-TAIL probabilities:**

$F_{.05}$

$F_{(1-\alpha),(n_1-1),(n_2-1)}$

$= 1/\underline{F_{(\alpha)}}, (n_2-1), (n_1-1)$ "interim" $F$

NOTE the reversal of the degrees of freedom

For Example:

Suppose $\alpha = 0.05$, $n_1 = 7$, $n_2 = 11$

So $F_{0.05, 6, 10} = 3.22$

To obtain $F_{0.95, 6, 10}$ :

Acquire $F_{0.05, 10, 6} = 4.06$

and then $F_{0.95, 6, 10} = \frac{1}{4.06} = 0.246$

Two sided test - Say $\alpha = 0.05$

$F_{\alpha/2, n_1-1, n_2-1}$ for upper limit

$1/F_{\alpha/2, n_2-1, n_1-1}$ for lower limit

# EXAMPLE

**It is desired to determine whether there is less variability in the silver plating done by Company 1 than in that done by Company 2. Independent random samples of size 12 yield standard deviations of 0.035 and 0.052 mil for Company 1 and Company 2, respectively. For $\alpha = 0.05$, are we willing to conclude that there is less variability in the silver plating done by Company 1?**

$H_0: \sigma_1^2 - \sigma_2^2 = 0$

$H_1: \sigma_1^2 - \sigma_2^2 < 0$

$\alpha = 0.05$

$n_1 = 12 \Rightarrow dof_1 = 11$

$n_2 = 12 \Rightarrow dof_2 = 11$

$S_1 = 0.035 \Rightarrow S_1^2 = 0.001225$

$S_2 = 0.052 \Rightarrow S_2^2 = 0.002704$

Test statistic: $F = S_1^2 / S_2^2 = \dfrac{0.001225}{0.002704} \doteq 0.453$

$F_{crit} = F_{(1-\alpha),(n_1-1),(n_2-1)} = F_{0.95,11,11}$

$F_{0.95,11,11} = 1/F_{0.05,11,11} = 1/2.82 \doteq 0.355$

Since $F_{test} > F_{crit}$, we cannot reject $H_0$