Laplace's Classical Definition of Mathematical Probability:

The probability P(A) of an event A in a random experiment is

$$P(A) = w / m,$$

where w is the number of cases in which A occurs and m is the number of all equally likely cases in the experiment.

"relative frequency" definition

Probability Concept in Statistics:

If we perform several lengthy replicates of a random experiment, the corresponding relative frequencies will be almost the same. We say that the experiment shows *statistical regularity* or *stability of the relative frequencies*.

For a particular random experiment, we postulate the existence of a number, *P(E)*, which is called the *probability of occurrence* of the event *E*. The statement *"E has the probability P(E)"* then means that if we perform the experiment very often, it is practically certain that the relative frequency of occurrence of *E* is approximately equal to *P(E)*.

The probability thus introduced is the counterpart of the empirical relative frequency.

Axioms of Mathematical Probability:

1. If E is any event in a sample space S, then

$$0 \leq P(E) \leq 1$$

2. To the entire sample space S there corresponds

$$P(S) = 1$$

3. If A and B are mutually exclusive events, then

$$P(A \cup B) = P(A) + P(B)$$

nothing in common

union

OR

Selected Basic Concepts:

- If A and B are any events in a sample space S, then
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$ — both can occur
"or" intersection "and"

- The probabilities of an event E and its complement E' in a sample space S are related by the formula
$$P(E) = 1 - P(E')$$
E' "not" $\overline{E}$

- If A and B are events in a sample space S and P(A) >0, P(B) > 0, then
given
$$P(A \cap B) = P(A) \cdot P(B|A) = P(B) \cdot P(A|B),$$
where P(A|B) is the conditional probability of A given B, and analogously for P(B|A)

- Two events A and B in a random experiment are said to be statistically independent events if
$$P(A \cap B) = P(A) \cdot P(B)$$

- Bayes' Rule:

If $A_1, A_2, \dots, A_j$ are j mutually exclusive and exhaustive events, and B is any event, then
$$P(A_k/B) = \frac{P(B/A_k) \cdot P(A_k)}{\sum_j P(B/A_j) \cdot P(A_j)}$$
for $P(B) > 0$

- **If A and B are any events in a sample space S, then**

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

<u>Example</u>:

A fair die is to be rolled once. You win if the outcome is either even or evenly divisible by 3. What is the probability of winning the game?

Let A = outcome is even

B = outcome is divisible by 3

$S = \{1, 2, 3, 4, 5, 6\}$; $A = \{2, 4, 6\}$; $B = \{3, 6\}$

$A \cap B = \{6\}$

we win

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= 3/6 + 2/6 - 1/6 = 2/3$

- **The probabilities of an event E and its complement E′ in a sample space S are related by the formula**

$$P(E) = 1 - P(E')$$

Example:

A game consists of rolling a fair die. You lose if your die displays the value 3. What is your probability of winning?

Let $P(\text{losing}) = P(3) = 1/6$

Then $P(\text{winning}) = 1 - 1/6 = 5/6$

- **If A and B are events in a sample space S and P(A) > 0, P(B) > 0, then**

$$P(A \cap B) = P(A) \cdot P(B|A) = P(B) \cdot P(A|B),$$

where P(A|B) is the conditional probability of A given B, and analogously for P(B|A)

Example:

Two cards are to be drawn in sequence and without replacement from a standard deck of playing cards. What is the probability that both cards are Kings?

There are 52 cards in the deck,
4 of which are Kings

Let $A \Rightarrow 1^{st}$ card is a King
$B \Rightarrow 2^{nd}$ card is a King

$P(A) = 4/52$

Given that the first card is a King, and that it is not replaced in the deck,

then $P(B/A) = 3/51$

Thus $P(A \cap B) = P(A) \cdot P(B/A)$

$= 4/52 \cdot 3/51 = 0.0045$

or approx 0.45%

- **Two events A and B in a random experiment are said to be statistically independent events if**

$$P(A \cap B) = P(A) \cdot P(B)$$

Example:

A fair coin is to be tossed twice in succession. What is the probability of obtaining two heads?

Since the outcome of the first toss has no effect on the outcome of the second toss, the events are independent

$P(A)$ = probability of head on first toss = $1/2$

$P(B)$ = probability of head on second toss = $1/2$

Thus $P(A \cap B) = P(A) \cdot P(B) = (1/2)(1/2) = 1/4$

- **Bayes' Rule (Theorem):**

If $A_1, A_2, \ldots, A_j$ are j mutually exclusive and exhaustive events, and B is any event, then

$$P(A_k/B) = \frac{P(B/A_k)\cdot P(A_k)}{\sum_j P(B/A_j)\cdot P(A_j)}$$

for $P(B) > 0$

Note that this can also be stated as

$$P(A_k/B) = \frac{P(A_k \cap B)}{\sum_j P(A_j \cap B)} \rightarrow \text{equivalent of } P(B)$$

where $P(A_j \cap B) = P(B \cap A_j)$

<u>**Example**</u>:

In a factory, employees work during three shifts. It has been observed over a long period of time that the first shift produces 35 percent, the second shift 30 percent, and the third shift 35 percent, of the total production. However, the number of defective items produced in each shift varies greatly, and are in the proportion 1:2:3, respectively. If articles produced by the three shifts are placed in one well-mixed pile and one item randomly selected proves defective, what is the probability that the defective item was produced by (a) the first shift? (b) the third shift?

Bayes' Rule (Theorem) - Continued:

Let $A_1 \Rightarrow$ an item is produced by the first shift
$A_2 \Rightarrow$ an item is produced by the second shift
$A_3 \Rightarrow$ an item is produced by the third shift
$B \Rightarrow$ an item is defective

Given $P(A_1) = 0.35$; $P(A_2) = 0.30$; $P(A_3) = 0.35$

Since the defective items produced are in the proportions 1 : 2 : 3, respectively, then

$$P(B/A_1) = 1/6 \ ; \quad P(B/A_2) = 2/6 \ ; \quad P(B/A_3) = 3/6$$

Thus we can determine that

Determining P(B)

$$P(B \cap A_1) = P(B/A_1) \cdot P(A_1) = \left(\frac{35}{100}\right)\left(\frac{1}{6}\right) = \frac{35}{600}$$

$$P(B \cap A_2) = P(B/A_2) \cdot P(A_2) = \left(\frac{30}{100}\right)\left(\frac{2}{6}\right) = \frac{60}{600}$$

$$P(B \cap A_3) = P(B/A_3) \cdot P(A_3) = \left(\frac{35}{100}\right)\left(\frac{3}{6}\right) = \frac{105}{600}$$

So that $\sum_j P(A_j \cap B) = \frac{35}{600} + \frac{60}{600} + \frac{105}{600} = \frac{1}{3} = P(B)$

Thus

(a) $P(A_1/B) = P(A_1 \cap B) \overset{\div}{/} P(B) = \left(\frac{35}{600}\right) \overset{\div}{/} \left(\frac{1}{3}\right) = \frac{35}{200} = 0.175$

(b) $P(A_3/B) = P(A_3 \cap B) / P(B) = \left(\frac{105}{600}\right) / \left(\frac{1}{3}\right) = \frac{105}{200} = 0.525$

$P(A_1 \cap B) = P(A_1 | B)$

$\Rightarrow P(A_1|B) = P(A_1 \cap B) \div P(B)$