SAMPLING DISTRIBUTIONS

Thus far have assumed knowledge of the population mean and variance. In general, they are not known.

μ σ^2

Usually selecting samples from a population in order to make inferences about the mean and variance.

also proportions

Regardless of the distribution of the population, the means of samples selected from a population are approximately normally distributed with mean μ and variance σ^2/n. (The square root of this variance is called the *standard error of the mean.*)

If the sample size is close to the population size, the variance is generally adjusted by the *finite population correction factor*, *(N-n)/(N-1)*.

When n is "large," the corresponding standardized random variate,

$$z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}}$$

is approximately Normally distributed.

STUDENT'S t-DISTRIBUTION

W.S. Gosset, 1908

Sampling distribution of the mean when the population variance cannot be assumed to be "known"

Holds for samples selected from normally distributed populations

The probability density function may be written as

$$f(t;k) = \frac{\Gamma\left(\frac{k+1}{2}\right)}{\sqrt{\pi k}\,\Gamma\left(\frac{k}{2}\right)}\left(1+\frac{t^2}{k}\right)^{-(k+1)/2}$$

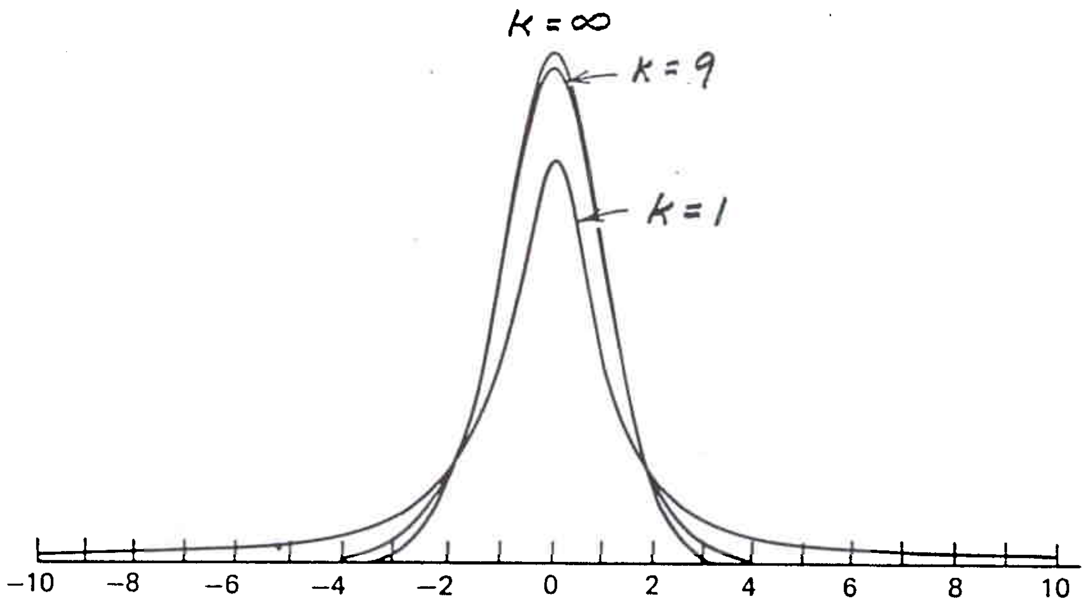
where $-\infty < t < \infty$

$k > 0$ is called the degrees of freedom

Γ is the Gamma function

STUDENT'S t-DISTRIBUTION

When *n* is "large," the Student's t-Distribution can be approximated by the Normal Distribution



when $k = \infty$, $P_r(t > 2) = 0.023$
$= 9$ $= 0.038$
$= 1$ $= 0.148$

The standardized variate for the t-Distribution is

$$t = \frac{\bar{x} - \mu}{s/\sqrt{n}}$$

with *k=n-1* degrees of freedom

STUDENT'S t-DISTRIBUTION

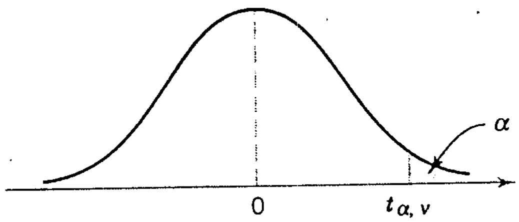


Table IV Percentage Points $t_{\alpha,\nu}$ of the t-Distribution

ν \ α	.40	.25	.10	.05	.025	.01	.005	.0025	.001
1	.325	1.000	3.078	6.314	12.706	31.821	63.657	127.32	318.31
2	.289	.816	1.886	2.920	4.303	6.965	9.925	14.089	23.32
3	.277	.765	1.638	2.353	3.182	4.541	5.841	7.453	10.2
4	.271	.741	1.533	2.132	2.776	3.747	4.604	5.598	7.
5	.267	.727	1.476	2.015	2.571	3.365	4.032	4.773	
6	.265	.718	1.440	1.943	2.447	3.143	3.707	4.317	
7	.263	.711	1.415	1.895	2.365	2.998	3.499	4.02	
8	.262	.706	1.397	1.860	2.306	2.896	3.355		
9	.261	.703	1.383	1.833	2.262	2.821	3.250		
10	.260	.700	1.372	1.812	2.228	2.764	3.		
11	.260	.697	1.363	1.796	2.201	2.718			
12	.259	.695	1.356	1.782	2.179	2.681			
13	.259	.694	1.350	1.771	2.160	2.650			
14	.258	.692	1.345	1.761	2.145	2.62			
15	.258	.691	1.341	1.753	2.131				
16	.258	.690	1.337	1.746	2.1				
17	.257	.689	1.333	1.74					
18	.257	.688	1.330						
19	.257	.688	1.32						
20	.257	.687	1.						
21	.257								
22									

INTERVAL ESTIMATES OF PARAMETERS

CONFIDENCE INTERVALS

Confidence Interval for the population mean, μ, of a Normal distribution with "known" variance, σ^2

"large" sample.

Let $\bar{x}$ = sample mean

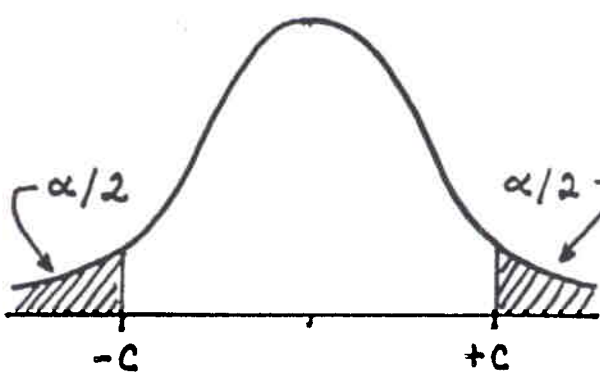
n = sample size

α = significance level we define

c = critical value = Normal distribution z-score corresponding to $\alpha/2$

Set $k = c\sigma/\sqrt{n}$

Then CONF $\{ (\bar{x} - k) \leq \mu \leq (\bar{x} + k) \}$



INTERVAL ESTIMATES OF PARAMETERS

CONFIDENCE INTERVALS

Confidence Interval for the population mean, μ, of a Normally distributed variate with "unknown" variance, s^2

small sample

Let $\bar{x}$ = sample mean

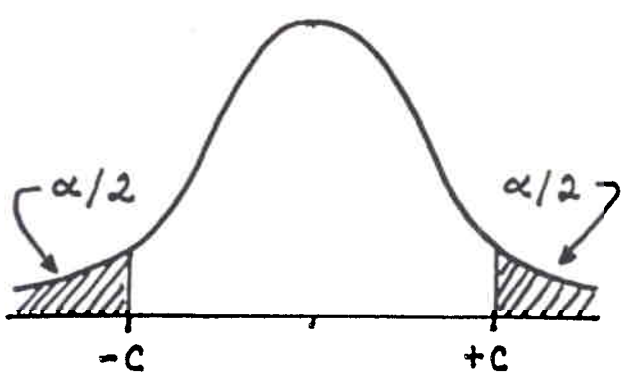
n = sample size

α = significance level

c = critical value = t-distribution score corresponding to $\alpha/2$ with $n-1$ degrees of freedom

Set $k = cs/\sqrt{n}$

Then CONF $\{ (\bar{x} - k) \leq \mu \leq (\bar{x} + k) \}$



EXAMPLE

The diameter of holes for cable harness is known to have a standard deviation of 0.01 inch. A random sample of size 10 yields an average diameter of 1.5045 inch.

(sample size makes no difference

Determine the 99% confidence interval for the mean hole diameter.

z_{crit} based on α

$\sigma = 0.01$
$\bar{x} = 1.5045$
$n = 10$

$\alpha = 0.01 \Rightarrow \alpha/2 = 0.005$
$c = 2.575$ Z score
2.58

$$k = c\sigma/\sqrt{n} = (2.575)(0.01)/\sqrt{10}$$
$$= 0.0081$$

$$\bar{x} - k = 1.5045 - 0.0081 = 1.4964$$
$$\bar{x} + k = 1.5045 + 0.0081 = 1.5126$$

Thus $CONF_{0.99}\{1.4964 \le \mu \le 1.5126\}$ inches

EXAMPLE

The wall thickness of 25 two-liter glass bottles was measured by a quality-control engineer. The sample mean was 4.05 millimeters, and the sample standard deviation was 0.08 millimeter.

Determine the 95% Confidence Interval for the mean.

$s = 0.08$ $\qquad \alpha = 0.05 \Rightarrow \alpha/2 = 0.025$

$\bar{x} = 4.05$ $\qquad dof = n-1 = 24$ t-distribution

$n = 25$ $\qquad c = 2.064$

$$k = cs/\sqrt{n} = (2.064)(0.08)/\sqrt{25}$$

$$\bar{x} - k = 4.05 - 0.03 = 4.02$$

$$\bar{x} + k = 4.05 + 0.03 = 4.08$$

Thus $CONF_{95\%}\{4.02 \le \mu \le 4.08\}$ millimeters

Hypothesis Testing
General Procedure

1. Identify the parameter of interest (e.g., the mean)
2. State the null hypothesis, H_0
3. State the alternative hypothesis, H_1
4. Identify the significance level, α
5. Identify the test statistic $\left\{ \text{e.g., } z = \dfrac{\bar{x} - \mu}{\sigma/\sqrt{n}} \right\}$
6. Determine the rejection region (based on H_1 and α) and the corresponding critical value(s) of the test statistic
7. Calculate sample values (e.g., $\bar{x}$) as may be required
8. Using the test statistic formula, the sample values, and the H_0 parameter value(s), calculate the test statistic value
9. Compare the test value from step 8 to the critical value from step 6 to determine if H_0 should be rejected
10. State the conclusion

HYPOTHESIS TESTING

POPULATION MEAN - "LARGE" SAMPLE

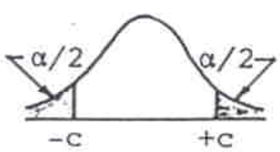
TEST OF HYPOTHESIS CONCERNING THE MEAN OF A POPULATION

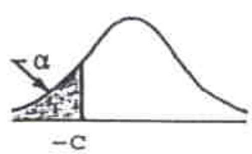
(for large samples; i.e., $n \gtrsim 30$)

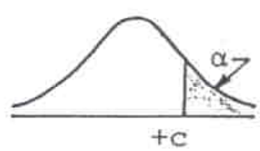
Test Statistic: $z = \dfrac{\overline{x} - \mu}{\hat{\sigma}_{\overline{x}}}$, where $\hat{\sigma}_{\overline{x}} = \dfrac{s}{\sqrt{n}}$

	CASE 1:	CASE 2:	CASE 3:
HYPOTHESES	$H_0: \ \mu = \mu_0$ $H_1: \ \mu \neq \mu_0$	$H_0: \ \mu = \mu_0$ $H_1: \ \mu < \mu_0$	$H_0: \ \mu = \mu_0$ $H_1: \ \mu > \mu_0$

REJECTION REGION(S)







	CASE 1	CASE 2	CASE 3
DECISION RULE	Reject H_0 if $\lvert z \rvert > \lvert c \rvert$ $z < -c$ or $z > +c$	Reject H_0 if $z < -c$	Reject H_0 if $z > c$

μ_0 is the value of the population mean specified in the hypotheses. c is the critical value obtained from Normal distribution tables for a particular α.

HYPOTHESIS TESTING

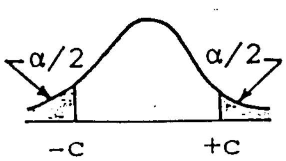
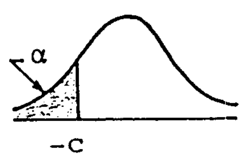
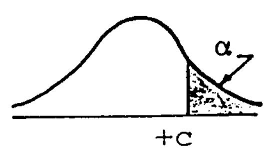
POPULATION MEAN - "SMALL" SAMPLE

TEST OF HYPOTHESIS CONCERNING THE MEAN OF A POPULATION

(for small samples; i.e., $n \lesssim 30$)

Test Statistic: $t = \dfrac{\bar{x} - \mu}{\hat{\sigma}_{\bar{x}}}$, where $\hat{\sigma}_{\bar{x}} = \dfrac{s}{\sqrt{n}}$,

with (n-1) degrees of freedom

	CASE 1:	CASE 2:	CASE 3:
HYPOTHESES	H_o: $\mu = \mu_o$ H_1: $\mu \neq \mu_o$	H_o: $\mu = \mu_o$ H_1: $\mu < \mu_o$	H_o: $\mu = \mu_o$ H_1: $\mu > \mu_o$
REJECTION REGION(S)			
DECISION RULE	Reject H_o if $\lvert t \rvert > \lvert c \rvert$	Reject H_o if $t < -c$	Reject H_o if $t > c$

$t < -c$ or $t > +c$

μ_o is the value of the population mean specified in the hypotheses. c is the critical value obtained from t-distribution tables for a particular α.

EXAMPLE

The mean weight of a tablet of a certain drug is supposed to be 50 milligrams. A random sample of 64 tablets has a mean weight of 50.15 milligrams and a standard deviation of 0.4 milligrams. Using a 0.01 level of significance, can we conclude that the desired weight of the tablet is not properly maintained?

$n = 64 \Rightarrow$ can use large-sample test

$H_0 : \mu = 50$ $\quad\quad \bar{x} = 50.15$ mg

$H_1 : \mu \neq 50$ $\quad\quad \hat{\sigma} = s = 0.4$ mg

$\alpha = 0.01$ and two-sided test $\Rightarrow$ use $\alpha/2 = 0.005$

large $n \Rightarrow$ Normal Table $\Rightarrow \overset{\mp}{c} = \pm 2.575$

Compute $z = \dfrac{\bar{x} - \mu}{\hat{\sigma}/\sqrt{n}} = \dfrac{50.15 - 50}{0.4/\sqrt{64}} = 3.0$

Since $(z = 3) > (c = 2.575)$, Reject H_0,

and conclude that the desired weight of the tablet is not being properly maintained

EXAMPLE

A manufacturer of a certain car model claims that the average mileage of this model is 30 miles per gallon of regular gasoline. A consumer protection agency believes that the average mileage of the car is exaggerated by the manufacturer. Nine cars of this particular model are driven in the same manner with one gallon of regular gasoline. The distances traveled by the different cars are

30, 28, 26, 27, 29, 28, 31, 26, and 27 miles.

If the agency is willing to reject a true claim no more than once in 100, would the agency reject the manufacturer's claim?

$n = 9$ and population variance unknown
$\Rightarrow$ use small-sample test

$H_0: \mu = 30$ $\quad\quad \bar{x} = 252/9 = 28$

$H_1: \mu < 30$ $\quad\quad s = \sqrt{\frac{24}{8}} = \sqrt{3}$

$\alpha = 0.01$ and one-sided test $\Rightarrow$ use $\alpha = 0.01$

small $n \Rightarrow t$ Table $\Rightarrow \underset{}{\overset{t\ \text{score}}{c}} = 2.896$ for 8 dof

must be negative due to direction of test

Compute $t = \dfrac{\bar{x} - \mu}{s/\sqrt{n}} = \dfrac{28 - 30}{\sqrt{3}/3} = -3.47$

Since $(t = |-3.47|) > (c = 2.896)$, Reject H_0

Since $(\underbrace{t = -3.47}_{\text{calculated}}) < (c = -2.896) \leftarrow$ table

Reject H_0.

HYPOTHESIS TESTING

Comparison of Two Means - Large Samples

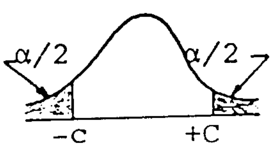
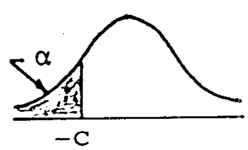
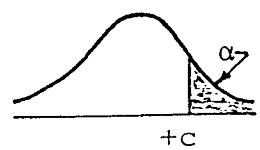
(population variances known)

TEST OF HYPOTHESIS CONCERNING THE MEANS OF TWO POPULATIONS

(when both samples are large; i.e., n_1, $n_2 \gtrsim 30$).

Test Statistic: $z = \dfrac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\hat{\sigma}_d}$, where $\hat{\sigma}_d = \sqrt{\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}}$

$H_0: \mu_1 = \mu_2$
$H_1: \mu_1 \neq \mu_2$

	CASE 1:	CASE 2:	CASE 3:
HYPOTHESES	$H_o: \mu_1 - \mu_2 = d$ $H_1: \mu_1 - \mu_2 \neq d$	$H_o: \mu_1 - \mu_2 = d$ $H_1: \mu_1 - \mu_2 < d$	$H_o: \mu_1 - \mu_2 = d$ $H_1: \mu_1 - \mu_2 > d$
REJECTION REGION(S)			
DECISION RULE	Reject H_o if $\lvert z \rvert > \lvert c \rvert$	Reject H_o if $z < -c$	Reject H_o if $z > c$

d is the difference between the two hypothesized population means and is most often taken to be zero. c is the critical value obtained from Normal distribution tables for a particular value of α.

HYPOTHESIS TESTING

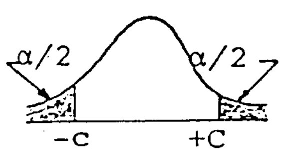
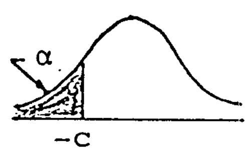
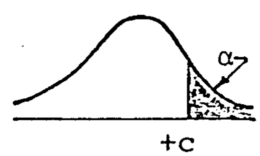
Comparison of Two Means - Small Samples
variance unknown

TEST OF HYPOTHESIS CONCERNING THE MEANS OF TWO POPULATIONS

(when both samples are small; i.e., n_1, $n_2 \lesssim 30$)

Test Statistic: $$t = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\hat{\sigma}_d}, \quad \hat{\sigma}_d = \sqrt{\frac{(n_1-1)s_1^2 + (n_2-1)s_2^2}{(n_1+n_2-2)}\left(\frac{1}{n_1}+\frac{1}{n_2}\right)},$$

with $(n_1 + n_2 - 2)$ degrees of freedom

	CASE 1:	CASE 2:	CASE 3:
HYPOTHESES	$H_o: \mu_1 - \mu_2 = d$ $H_1: \mu_1 - \mu_2 \neq d$	$H_o: \mu_1 - \mu_2 = d$ $H_1: \mu_1 - \mu_2 < d$	$H_o: \mu_1 - \mu_2 = d$ $H_1: \mu_1 - \mu_2 > d$
REJECTION REGION(S)			
DECISION RULE	Reject H_o if $\lvert t\rvert > \lvert c\rvert$	Reject H_o if $t < -c$	Reject H_o if $t > c$

d is the difference between the two hypothesized population means and is most often taken to be zero. c is the critical value obtained from Student t distribution tables for a particular value of α.

EXAMPLE

A simple random sample of 100 students attending University A resulted in an average age of 23 years and a standard deviation of 4 years, while a simple random sample of 50 students attending University B revealed an average age of 21 years and a standard deviation of 5 years. From the results of these two samples, can we safely conclude that the average age in the two universities is not the same?

Both samples are "large"

$H_0: \mu_A = \mu_B$ $\qquad \bar{x}_A = 23 \qquad \bar{x}_B = 21$

$\qquad s_A = 4 \qquad s_B = 5$

$H_1: \mu_A \neq \mu_B$ $\qquad n_A = 100 \qquad n_B = 50$

α not given; Let $\alpha = 0.05$; two-tailed test

Calculate $\hat{\sigma}_d = \sqrt{\frac{s_A^2}{n_A} + \frac{s_B^2}{n_B}} = 0.812$

$$z = \frac{(\bar{x}_A - \bar{x}_B) - (\mu_A - \mu_B)}{\hat{\sigma}_d} = \frac{(23-21)-0}{0.812} = 2.46$$

For $\alpha/2 = 0.025$, $\underset{c}{z} = 1.96$ from table

Since $(z = 2.46) > (c = 1.96)$, Reject H_0

EXAMPLE

To compare the vitamin A content of two different brands of vitamin capsules, a sample of 6 capsules was selected from each brand. The sample of Brand X had a mean of 5000 units and a standard deviation of 400 units. The sample of Brand Y had a mean of 4800 units and a standard deviation of 300 units. If you are willing to reject a true hypothesis no more than once in 100, can you conclude that the vitamin A contents of the two brands of capsules are not the same?

Both samples have "small" values of n

$H_0: \mu_x = \mu_y$ $\quad\bar{x}_x = 5000 \quad \bar{x}_y = 4800$

$s_x = 400 \quad s_y = 300$

$H_1: \mu_x \neq \mu_y$ $\quad n_x = 6 \quad n_y = 6$

$\alpha = 0.01$ and two-tailed test

$\Rightarrow \alpha/2 = 0.005$ with $(n_x + n_y - 2) = 10$ dof

Determine

$$\hat{\sigma}_d = \sqrt{\frac{(n_x-1)s_x^2 + (n_y-1)s_y^2}{(n_x+n_y-2)}} \cdot \sqrt{\frac{1}{n_x} + \frac{1}{n_y}} = 204.12$$

$$t = \frac{(\bar{x}_x - \bar{x}_y) - (\mu_x - \mu_y)}{\hat{\sigma}_d} = \frac{(5000-4800) - 0}{204.12} = 0.98$$

$\alpha/2 = 0.005$ and 10 dof $\Rightarrow \overset{t\ \pm}{c} = 3.169$

Since $(t = 0.98) \not> (c = 3.169)$

Do not reject H_0

ERRORS
and
OPERATING CHARACTERISTIC CURVES

Type I Error: Rejecting H_0 when H_0 is TRUE

α = Probability of Type I Error

Type II Error: Not rejecting H_0 when H_0 is FALSE

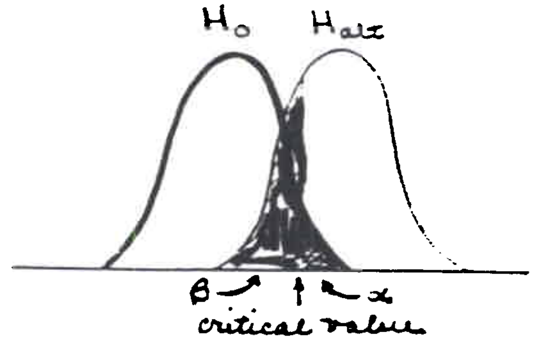
β = Probability of Type II Error

α = Significance Level

$1 - \beta$ = Power

α – Selected by statistician

β – Dependent on sample size



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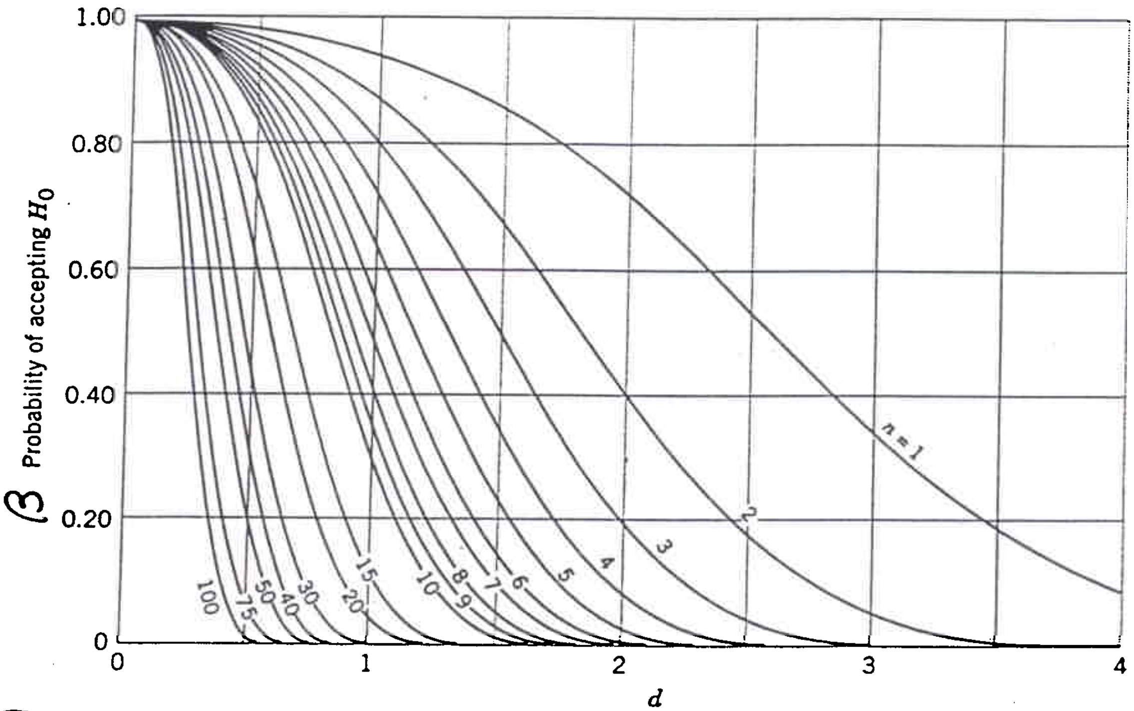
ERRORS
and
OPERATING CHARACTERISTIC CURVES



(b) *OC* curves for different values of *n* for the two-sided normal test for a level of significance $\alpha = 0.01$.

$$d = \frac{|\mu - \mu_0|}{\sigma}$$