


# Structural Equation Modeling



Continued: Lecture 2

Psy 524

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# Covariance Algebra

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- Underlying parameters in SEM
  - Regression Coefficients
  - Variances and Covariances
- A hypothesized model is used to estimate these parameters for the population (assuming the model is true)
  - The parameter estimates are used to create a hypothesized variance/covariance matrix
  - The sample VC matrix is compared to the estimated VC

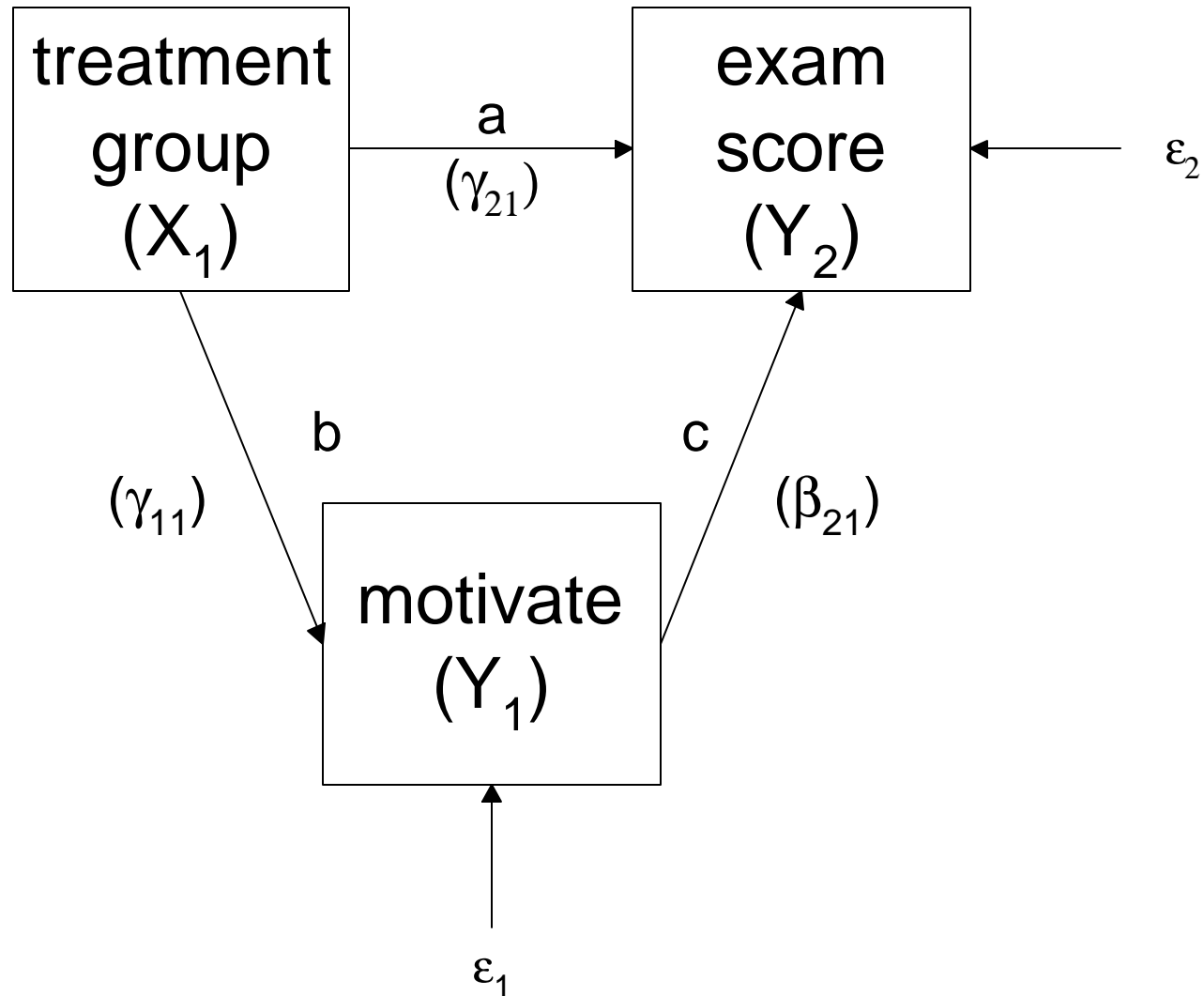
# Covariance Algebra

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- Three basic rules of covariance algebra
  - $\text{COV}(c, X_1) = 0$
  - $\text{COV}(cX_1, X_2) = c * \text{COV}(X_1, X_2)$
  - $\text{COV}(X_1 + X_2, X_3) = \text{COV}(X_1, X_3) + \text{COV}(X_2, X_3)$
  
- Reminder: Regular old fashioned regression
  - $Y = bX + a$

# Example Model

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# Covariance Algebra

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- In covariance structure intercept is not used
- Given the example model here are the equations
  - $Y_1 = \gamma_{11}X_1 + \varepsilon_1$ 
    - Here Y1 is predicted by X1 and e1 only
    - X1 is exogenous so  $\gamma$  is used to indicate the weight
  - $Y_2 = \beta_{21}Y_1 + \gamma_{21}X_1 + \varepsilon_2$ 
    - Y2 is predicted by X1 and Y1 (plus error)
    - Y1 is endogenous so the weight is indicated by  $\beta$
    - The two different weights help to indicate what type of relationship (DV, IV or DV, DV)

# Covariance Algebra

## Estimated Covariance given the model

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- $\text{Cov}(X_1, Y_1)$  the simple path
  - Since  $Y_1 = \gamma_{11}X_1 + \varepsilon_1$  we can substitute
  - $\text{COV}(X_1, Y_1) = \text{Cov}(X_1, \gamma_{11}X_1 + \varepsilon_1)$
  - Using rule 3 we distribute:
    - $\text{COV}(X_1, \gamma_{11}X_1 + \varepsilon_1) = \text{Cov}(X_1, \gamma_{11}X_1) + (X_1, \varepsilon_1)$
    - By definition of regression  $(X_1, \varepsilon_1) = 0$
    - So this simplifies to:  
$$\text{Cov}(X_1, \gamma_{11}X_1 + \varepsilon_1) = \text{Cov}(X_1, \gamma_{11}X_1)$$
  - Using rule 2 we can pull  $\gamma_{11}$  out:
    - $\text{COV}(X_1, Y_1) = \gamma_{11}\text{Cov}(X_1, X_1)$
    - $\text{COV}(X_1, X_1)$  is the variance of  $X_1$
    - So,  $\text{Cov}(X_1, Y_1) = \gamma_{11}\sigma_{x_1x_1}$

# Covariance Algebra

## Estimated Covariance given the model

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### □ $\text{COV}(Y_1, Y_2)$ the complex path

- Substituting the equations in for  $Y_1$  and  $Y_2$

- $\text{COV}(Y_1, Y_2) = \text{COV}(\gamma_{11}X_1 + \varepsilon_1, \beta_{21}Y_1 + \gamma_{21}X_1 + \varepsilon_2)$

- Distributing all of the pieces

- $\text{COV}(Y_1, Y_2) = \text{COV}(\gamma_{11}X_1, \beta_{21}Y_1) + \text{COV}(\gamma_{11}X_1, \gamma_{21}X_1) + \text{COV}(\gamma_{11}X_1, \varepsilon_2) + \text{COV}(\varepsilon_1, \beta_{21}Y_1) + \text{COV}(\varepsilon_1, \gamma_{21}X_1) + \text{COV}(\varepsilon_1, \varepsilon_2)$

- Nothing should correlate with  $e$  so they all drop out

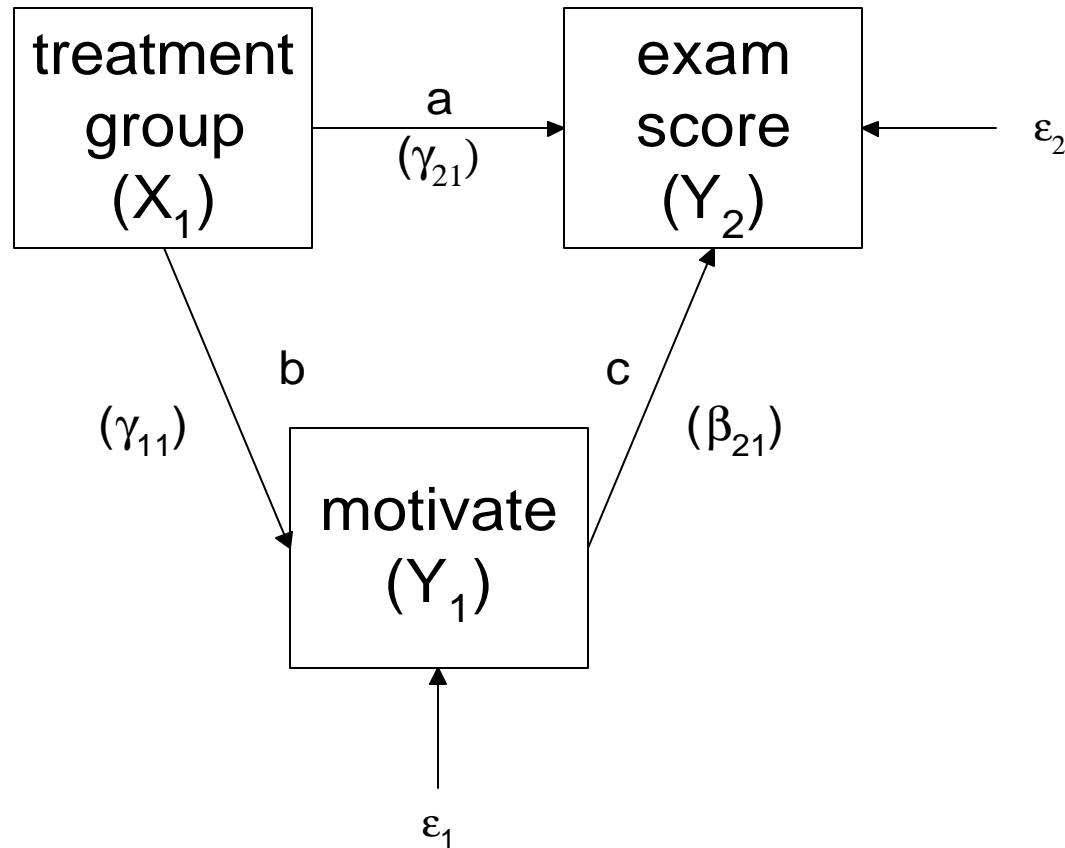
- $\text{COV}(Y_1, Y_2) = \text{COV}(\gamma_{11}X_1, \beta_{21}Y_1) + \text{COV}(\gamma_{11}X_1, \gamma_{21}X_1)$

- Rearranging:

- $\text{COV}(Y_1, Y_2) = \gamma_{11}\beta_{21}\text{COV}(X_1, Y_1) + \gamma_{11}\gamma_{21}\text{COV}(X_1, X_1)$

- $\text{COV}(Y_1, Y_2) = \gamma_{11}\beta_{21}\sigma_{x_1y_1} + \gamma_{11}\gamma_{21}\sigma_{x_1x_1}$

# Back to the path model



□  $\text{COV}(X_1, Y_1) = \gamma_{11} \sigma_{x_1}^2$

□  $\text{COV}(Y_1, Y_2) = \gamma_{11} \beta_{21} \sigma_{x_1 y_1} + \gamma_{11} \gamma_{21} \sigma_{x_1}^2$

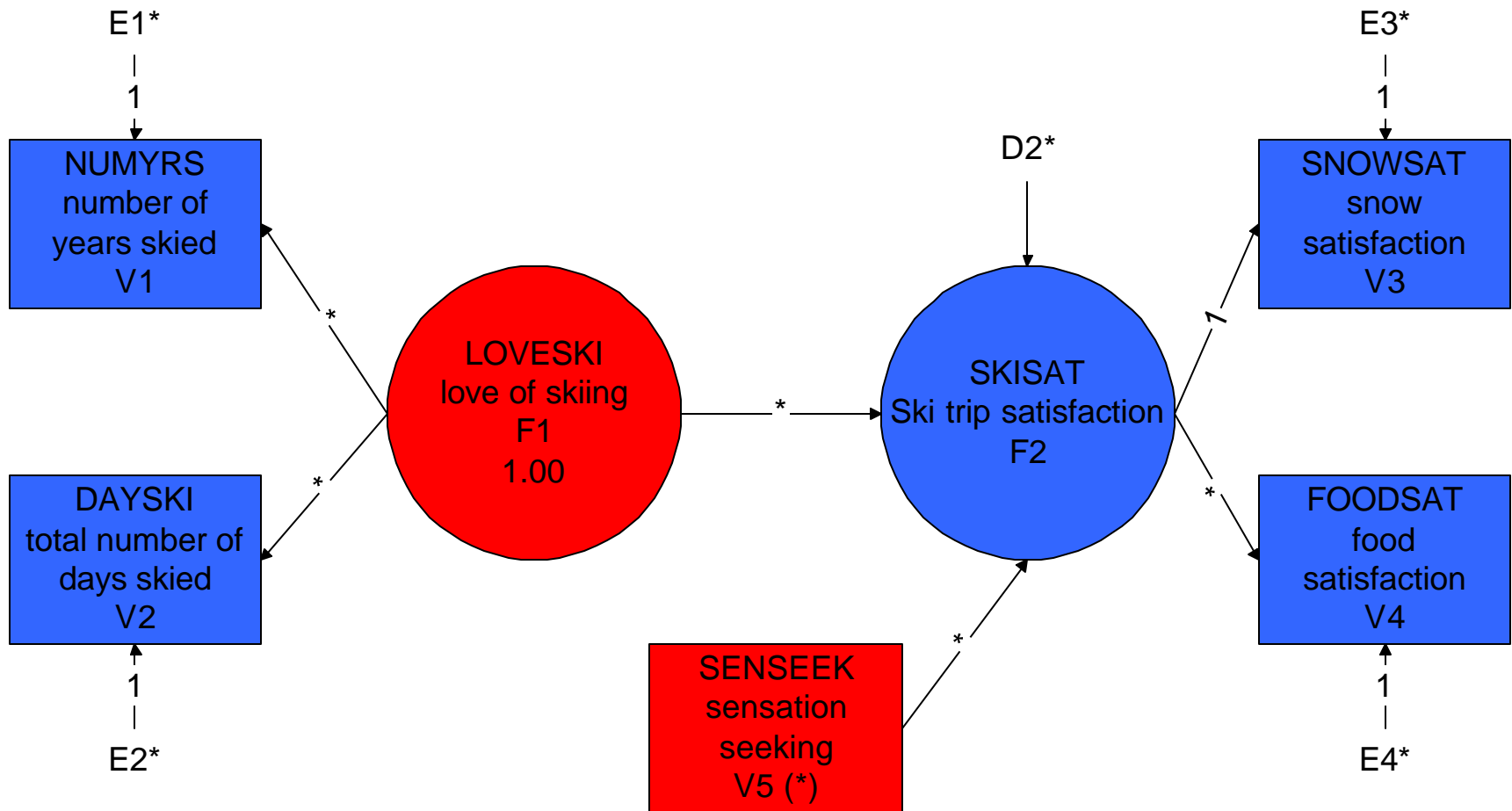
# Example

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Covariance Matrix

	NUMYRS	DAYSKI	SNOWSAT	FOODSAT	SENSEEK
NUMYRS	2.74	0.80	0.68	0.65	2.02
DAYSKI	0.80	3.25	0.28	0.35	2.12
SNOWSAT	0.68	0.28	1.23	0.72	2.02
FOODSAT	0.65	0.35	0.72	1.87	2.12
SENSEEK	2.02	2.12	2.02	2.12	27.00

# Example



# SEM models

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- All of the relationships in the model are translatable into equations
- Analysis in SEM is preceded by specifying a model like in the previous slide
- This model is used to create an estimated covariance matrix

# SEM models

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- The goal is to specify a model with an estimated covariance matrix that is not significantly different from the sample covariance matrix
- CFA differs from EFA in that the difference can be tested using a Chi-square test
  - If ML methods are used in EFA a chi-square test can be estimated as well

# Model Specification

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## □ Bentler-Weeks model

### ■ Matrices

- B – Beta matrix, matrix of regression coefficients of DVs predicting other DVs
- $\gamma$  – Gamma matrix, matrix of regression coefficients of DVs predicted by IVs
- $\Phi$  – phi matrix, matrix of covariances among the IVs
- $\eta$  – eta matrix, vector of DVs
- $\xi$  – xi matrix, vector of IVs

### ■ Bentler-Weeks regression model

$$\mathbf{h} = \mathbf{B} \mathbf{h} + \mathbf{g} \mathbf{x}$$

$q \times 1$        $q \times q$   $q \times 1$        $q \times r$   $r \times 1$

where q is number of DVs and r is number of IVs

# Model Specification

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- Bentler-Weeks model
  - Only independent variables have covariances (phi matrix)
  - The estimated parameters are in the  $B$ ,  $\gamma$  and  $\Phi$  matrices

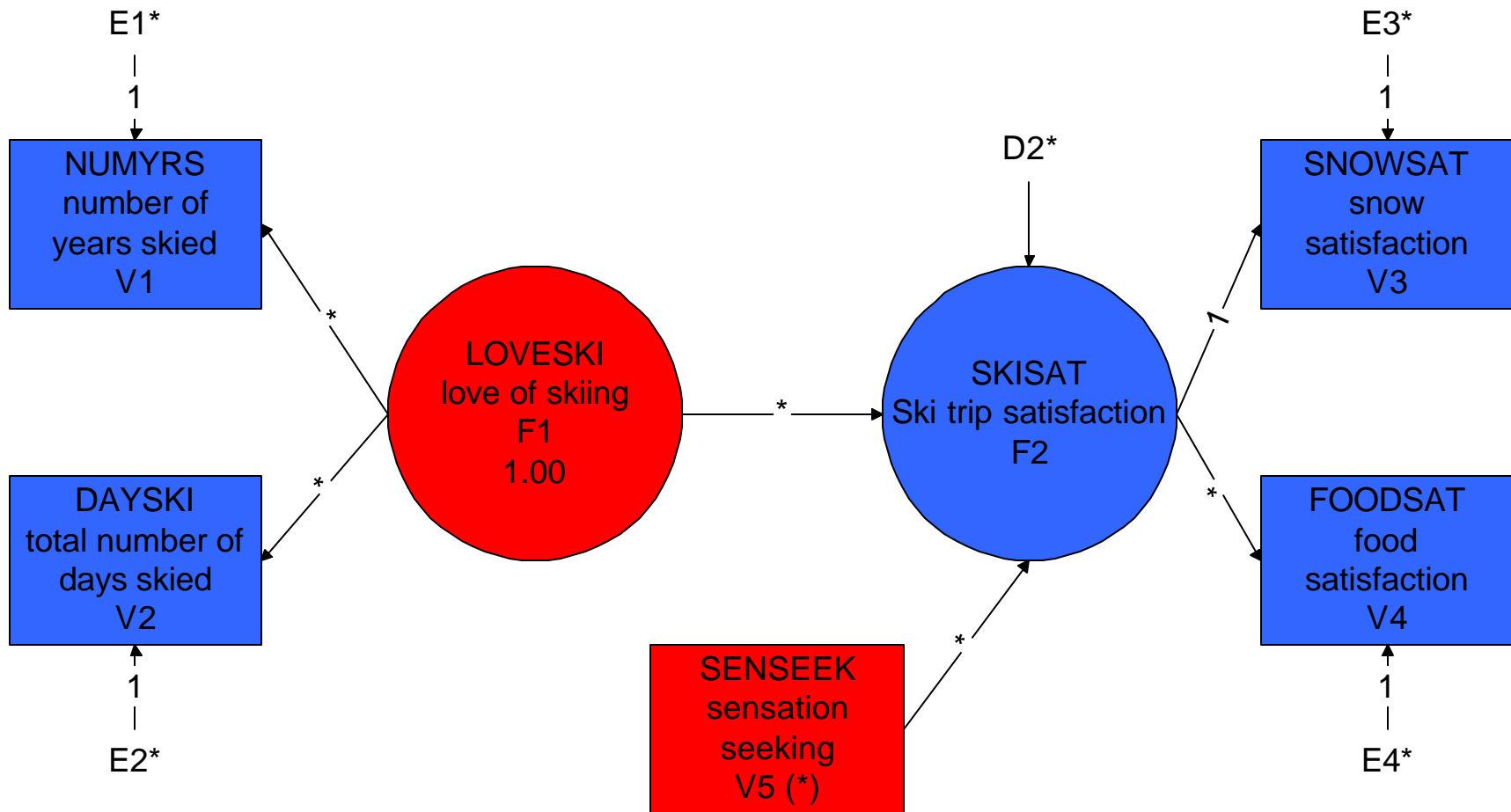
# Model Specification

- Bentler-Weeks model matrix form

$$\eta = B \eta + \gamma + \xi$$

$$\begin{bmatrix} V1 \text{ or } \eta_1 \\ V2 \text{ or } \eta_2 \\ V3 \text{ or } \eta_3 \\ V4 \text{ or } \eta_4 \\ F2 \text{ or } \eta_5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} V1 \text{ or } \eta_1 \\ V2 \text{ or } \eta_2 \\ V3 \text{ or } \eta_3 \\ V4 \text{ or } \eta_4 \\ F2 \text{ or } \eta_5 \end{bmatrix} + \begin{bmatrix} 0 & * & 1 & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ * & * & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} V5 \text{ or } \xi_1 \\ F1 \text{ or } \xi_2 \\ E1 \text{ or } \xi_3 \\ E2 \text{ or } \xi_3 \\ E3 \text{ or } \xi_4 \\ E4 \text{ or } \xi_5 \\ D2 \text{ or } \xi_6 \end{bmatrix}$$

# Diagram again



# Model Specification

## □ Phi Matrix

$$\Phi = \begin{array}{l} V5 \text{ or } \xi_1 \\ F1 \text{ or } \xi_2 \\ E1 \text{ or } \xi_3 \\ E2 \text{ or } \xi_4 \\ E3 \text{ or } \xi_5 \\ E4 \text{ or } \xi_6 \\ D2 \text{ or } \xi_7 \end{array} \begin{array}{c} V5 \text{ or } \xi_1 \\ F1 \text{ or } \xi_2 \\ E1 \text{ or } \xi_3 \\ E2 \text{ or } \xi_4 \\ E3 \text{ or } \xi_5 \\ E4 \text{ or } \xi_6 \\ D2 \text{ or } \xi_7 \end{array} \begin{bmatrix} * & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & * \end{bmatrix}$$

- It is here that other covariances can be specified

# Model Estimation

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- Model estimation in SEM requires start values
  - There are methods for generating good start values
  - Good means less iterations needed to estimate the model
  - Just rely on EQS or other programs to generate them for you (they do a pretty good job)

# Model Estimation

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- These start values are used in the first round of model estimation by inserting them into the Bentler-Weeks model matrices (indicated by a "hat") in place of the \*s  
 $\hat{\mathbf{B}}$ ,  $\hat{\mathbf{g}}$  and  $\hat{\mathbf{\Phi}}$  which indicate estimated matrices
- Using EQS we get start values for the example

$$\hat{\mathbf{B}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.00 \\ 0 & 0 & 0 & 0 & 1.00 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

# Model Estimation

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$$\hat{\gamma} = \begin{bmatrix} 0 & 1.13 & 1.00 & 0 & 0 & 0 & 0 \\ 0 & 0.67 & 0 & 1.00 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.00 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.00 & 0 \\ 0.04 & 1.48 & 0 & 0 & 0 & 0 & 1.00 \end{bmatrix}$$

$$\hat{\Phi} = \begin{bmatrix} 27.00 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.00 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.48 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2.91 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.62 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.04 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 27.54 \end{bmatrix}$$

# Model Estimation

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- Selection Matrices (G)
- These are used to pull apart variables to use them in matrix equations to estimate the covariances

$$\mathbf{G}_y = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

- So that  $Y = G_y * \eta$  and Y is only measured dependent variables

# Model Estimation

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□  $G_x = [1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0]$  so that:

$X = G_x * \xi$  and  $X$  is (are) the measured variable(s)

□ Rewriting the matrix equation to solve for  $\eta$  we get:

$$\eta = (I - \beta)^{-1} \gamma \xi$$

- This expresses the DVs as linear combinations of the IVs

# Model Estimation

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- The estimated population covariance matrix for the DVs is found by:

$$\hat{\Sigma}_{yy} = G_y (I - \hat{B})^{-1} \hat{g} \hat{\Phi} \hat{g}' (I - \hat{B})^{-1} G_y'$$

$$\hat{\Sigma}_{yy} = \begin{bmatrix} 2.76 & 0.76 & -1.68 & -1.68 \\ 0.76 & 3.36 & -0.99 & -1.00 \\ -1.68 & -0.99 & 30.41 & 29.91 \\ -1.68 & -1.00 & 29.91 & 31.06 \end{bmatrix}$$

# Model Estimation

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- Estimated covariance matrix between DVs and IVs

$$\hat{\Sigma}_{yx} = G_x (I - \hat{B})^{-1} \hat{\Phi} G_x'$$

$$\hat{\Sigma}_{yx} = \begin{vmatrix} 0.00 \\ 0.00 \\ .19 \\ .19 \end{vmatrix}$$

# Model Estimation

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- The covariance(s) between IVs is estimated by :

$$\widehat{\Sigma}_{xx} = G_x \widehat{\Phi} G_x'$$

$$\widehat{\Sigma}_{xx} = 27$$

# Model Estimation

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- Programs like EQS usually estimate all parameters simultaneously giving an estimated covariance matrix  $\hat{\sigma}$
- $S$  is the sample covariance matrix
- The residual matrix is found by subtracting  $\hat{\sigma}$  from  $S$
- This whole process is iterative so that after the first iteration the values of the hat matrices output are input as new starting values

# Model Estimation

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- Iterations continue until the function (usually ML) is minimized, this is called convergence
- After five iterations the residual matrix in the example is:

$$\text{Residual Covariance Matrix} = \begin{bmatrix} 0.00 & 0.00 & 0.14 & 0.12 & 2.02 \\ 0.00 & 0.00 & 0.12 & 0.19 & 2.12 \\ 0.14 & 0.12 & 0.05 & 0.05 & 0.39 \\ 0.12 & 0.19 & 0.05 & 0.05 & 0.49 \\ 2.02 & 2.12 & 0.39 & 0.49 & 0.00 \end{bmatrix}$$

# Model Estimation

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- When the model converges you get estimates of all of the parameters and a converged residual matrix

$$\hat{\mathbf{B}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.00 \\ 0 & 0 & 0 & 0 & 1.00 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

# Model Estimation

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$$\hat{\gamma} = \begin{bmatrix} 0 & 1.59 & 1.00 & 0 & 0 & 0 & 0 \\ 0 & 0.50 & 0 & 1.00 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.00 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.00 & 0 \\ 0.06 & 0.34 & 0 & 0 & 0 & 0 & 1.00 \end{bmatrix}$$

$$\hat{\Phi} = \begin{bmatrix} 27.0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.00 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.20 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3.00 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.50 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.15 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.45 \end{bmatrix}$$

$$\hat{\Sigma} = \begin{bmatrix} 2.74 & 0.80 & 0.55 & 0.55 & 0.00 \\ 0.80 & 3.25 & 0.17 & 0.17 & 0.00 \\ 0.55 & 0.17 & 1.18 & 0.67 & 1.64 \\ 0.55 & 0.17 & 0.67 & 1.82 & 1.64 \\ 0.00 & 0.00 & 1.64 & 1.64 & 27.00 \end{bmatrix}$$

$$\text{Residual} = \begin{bmatrix} 0.00 & 0.00 & 0.13 & 0.10 & 2.02 \\ 0.00 & 0.00 & 0.11 & 0.18 & 2.12 \\ 0.13 & 0.11 & 0.05 & 0.05 & 0.38 \\ 0.10 & 0.18 & 0.05 & 0.05 & 0.48 \\ 2.02 & 2.12 & 0.38 & 0.48 & 0.00 \end{bmatrix}$$

# Model Evaluation

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- $\chi^2$  is based in the function minimum (from ML estimation) when the model converges
  - The minimum value is multiplied by  $N - 1$
  - From EQS:  
 $(.08924)(99) = 8.835$
  - The DFs are calculated as the difference between the number of data points  $(p(p + 1)/2)$  and the number of parameters estimated
  - In the example  $5(6)/2 = 15$ , and there are 11 estimates leaving 4 degrees of freedom

# Model Estimation

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- $\chi^2(4) = 8.835, p = .065$
- The goal in SEM is to get a non-significant chi-square because that means there is little difference between the hypothesized and sample covariance matrices
- Even with non-significant model you need to test significance of predictors
  - Each parameter is divided by its SE to get a Z-score which can be evaluated
  - SE values are best left to EQS to estimate

# Final Model

- Final example model with (unstandardized) and standardized values

