

# INITIAL COMPLEX ASSOCIATED TO A JET SCHEME OF A DETERMINANTAL VARIETY

BOYAN JONOV

ABSTRACT. We show in this paper that the principal component of the first order jet scheme over the classical determinantal variety of  $m \times n$  matrices of rank at most 1 is arithmetically Cohen-Macaulay, by showing that an associated Stanley-Reisner simplicial complex is shellable.

## 1. INTRODUCTION

Let  $F$  be an algebraically closed field and  $\mathbf{A}_F^k$  the affine space of dimension  $k$  over  $F$ . By a variety in  $\mathbf{A}_F^k$  we will mean the zero set of a collection of polynomials over  $F$  in  $k$  variables; in particular, our varieties are not assumed irreducible. In [4] and [5], Košir and Sethuraman had studied jet schemes over classical determinantal varieties, and had described their components in a large number of cases. In particular, they had shown that the variety of first-order jets, or loosely the “algebraic tangent bundle,” over the determinantal variety of  $m \times n$  matrices ( $m \leq n$ ) of rank at most 1 has two components when  $m \geq 3$ . One component is simply the affine space  $\mathbf{A}_F^{mn}$  supported over the origin. The other component, which is much more interesting, is the closure of the set of tangents at the nonsingular points of the base determinantal variety. We denote this component by  $Y$ , and refer to it as the *principal component*. (When  $m = 2$ , the variety of first-order jets is irreducible, and coincides with the principal component  $Y$ .) The goal of this paper is to show that  $Y$  is arithmetically Cohen-Macaulay, i.e., its coordinate ring is Cohen-Macaulay.

Consider the truncated polynomial ring  $F[t]/(t^2)$ , and let  $X(t) = (f_{i,j}(t))_{i,j}$  be the generic  $m \times n$  ( $m \leq n$ ) matrix over this ring; thus, the  $(i, j)$  entry of  $X(t)$  is of the form  $f_{i,j}(t) = x_{i,j} + y_{i,j}t$ , where  $1 \leq i \leq m$ ,  $1 \leq j \leq n$ , and  $x_{i,j}, y_{i,j}$  are variables. Let  $I$  be the ideal of  $R = F[x_{i,j}, y_{i,j}]$ ,  $1 \leq i \leq m, 1 \leq j \leq n$ , generated by the coefficients of powers of  $t$  in each  $2 \times 2$  minor of the generic matrix  $X(t)$ . Then the variety of first-order jets over the  $m \times n$  matrices ( $m \leq n$ ) of rank at most 1 is precisely the zero set of  $I$ . Let  $J$  be the ideal of the principal component  $Y$ . In [5], Košir and Sethuraman showed that  $I$  is radical, and further, determined a Groebner basis for both  $I$  and  $J$  for the graded reverse lexicographical order using the following scheme:  $y_{1,1} > y_{1,2} > \cdots > y_{1,n} > y_{2,1} > \cdots > y_{2,n} > \cdots > y_{m,n} > x_{1,1} > x_{1,2} > \cdots > x_{1,n} > x_{2,1} > \cdots > x_{2,n} > \cdots > x_{m,n}$  (see [5, Theorem 2.4], [5, Proposition 3.3], as also [5, Remark 2.2]).

It follows easily from the description in [5, Theorem 2.4] of the Groebner basis  $G$  of  $J$  that the leading term ideal of  $J$ ,  $LT(J) := \langle lm(g); g \in G \rangle$ , is generated by the following family of monomials:

**Proposition 1.1.** (*Generators of  $LT(J)$* ) *The following families of monomials generate  $LT(J)$ :  $A = \{x_{i,l}x_{j,k} \mid 1 \leq i < j \leq m, 1 \leq k < l \leq n\}$ ,  $B = \{x_{i,k}y_{j,l} \mid 1 \leq i < j \leq m, 1 \leq k < l \leq n\}$ ,  $C = \{x_{k,p}y_{j,q}y_{i,r} \mid 1 \leq i < j \leq k \leq m, 1 \leq p < q < r \leq n\}$ ,  $D = \{x_{i,r}y_{j,q}y_{k,p} \mid 1 \leq i < j < k \leq m, 1 \leq p < q \leq r \leq n\}$ , and  $E = \{y_{i,r}y_{j,q}y_{k,p} \mid 1 \leq i < j < k \leq m, 1 \leq p < q < r \leq n\}$ .*

Since  $LT(J)$  is generated by squarefree monomials we can construct the Stanley-Reisner complex  $\Delta_{LT(J)}$  of  $LT(J)$ : this is the simplicial complex on vertices  $\{x_{i,j}, y_{i,j} \mid 1 \leq i \leq m, 1 \leq j \leq n\}$  whose corresponding Stanley-Reisner ideal (see [6, Chap. 1] or [1, Chap. 5]) is  $LT(J)$ . The simplicial complex is defined by the relation  $x_{i_1,j_1} \cdots x_{i_k,j_k} \cdots x_{i_s,j_s} y_{i'_1,j'_1} \cdots y_{i'_l,j'_l} \cdots y_{i'_t,j'_t}$ ,  $1 \leq i_k, i'_l \leq m, 1 \leq j_k, j'_l \leq n$ , is a face of  $\Delta_{LT(J)}$  if, as a monomial, it does not belong to  $LT(J)$ .

We will enumerate all the facets of  $\Delta_{LT(J)}$  and we will describe an explicit ordering of the facets which will show that  $\Delta_{LT(J)}$  is a shellable simplicial complex. By standard results, shellability of  $\Delta_{LT(J)}$  allows us to conclude the following main result of the paper (a result that has been independently obtained by Smith and Weyman in [7] as well, using their geometric technique for computing syzygies):

**Theorem 1.2.** *The coordinate ring of  $Y$ , i.e.,  $R/J$ , is Cohen-Macaulay.*

We wish to thank Professor Aldo Conca for some very valuable discussions during the writing of the paper. This paper constitutes our M.S. thesis at California State University Northridge, and we wish to thank Professor B.A. Sethuraman for suggesting this problem and for his constant encouragement.

## 2. DESCRIBING THE FACETS OF $\Delta_{LT(J)}$

It would be helpful in what follows to visualize the structure of the monomials in the families  $A$ ,  $B$ ,  $C$ ,  $D$ , and  $E$  as described in Proposition 1.1. For this, see Figure 1. In this paper, we will visualize a monomial as being positioned in a matrix, where each variable of the monomial is located in the matrix's entry corresponding to the index of the variable.

In this section, we will enumerate all facets of  $\Delta_{LT(J)}$ . First, some notation: we will denote a facet  $F$  of the simplicial complex  $\Delta_{LT(J)}$  by  $F = F_x F_y$ , where  $F_x$  is a string composed of vertices  $x_{i,j}$ 's and  $F_y$  is a string composed of vertices  $y_{i,j}$ 's. We will view each of  $F_x$  and  $F_y$  as both strings of vertices or monomials, depending on the context. Note that  $F_x F_y \in \Delta_{LT(J)}$  if, as a monomial,  $F_x F_y$  does not belong in the ideal  $LT(J)$  if and only if  $F_x F_y$  is not divisible by the generators of  $LT(J)$  (see Proposition 1.1).

We will start by showing a relation between the facets of  $\Delta_{LT(J)}$  and those of the corresponding simplicial complexes arising from classical determinantal varieties. We refer to the excellent survey paper of Bruns and Conca

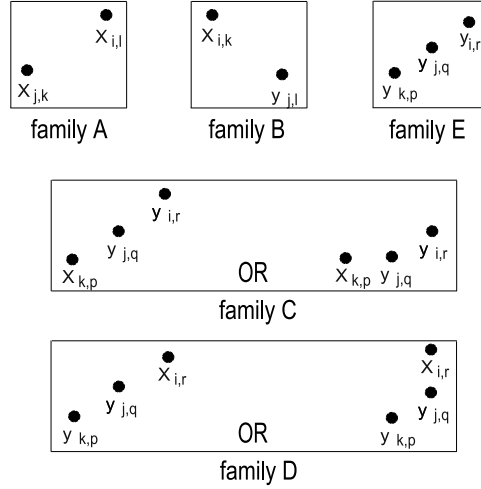


FIGURE 1.

[8]. In this paper, the authors consider the facets of  $\Delta_t$ : the Stanley-Reisner complex attached to the ideal  $LT(I_t)$ , which is generated by the leading terms of the  $t \times t$  minors of the generic  $m \times n$  matrix  $(w_{i,j})$ . The order they use is one in which the leading term of a minor is the main diagonal, and it is known that the leading terms of the  $t \times t$  minors generate the ideal of leading terms of  $I_t$ .

The key result for us is [8, Prop. 6.4], where they enumerate the facets of  $\Delta_t$ . This is a purely combinatorial result that enumerates the maximal subsets of  $V = \{w_{i,j} : i \leq m, j \leq n\}$  that intersect any  $t$ -subset of  $V$  arising from the diagonal of some  $t \times t$  submatrix of  $(w_{i,j})$  in at most  $t - 1$  places, and can be applied by symmetry to enumerate the maximal subsets of  $V$  that intersect any  $t$ -subset of  $V$  arising from the antidiagonal of some  $t \times t$  submatrix of  $(w_{i,j})$  in at most  $t - 1$  places. We quote this result as:

**Proposition 2.1.** ([8, Prop. 6.4]). *Let  $I_t$  be the ideal of  $F[\{w_{i,j}\}]$  generated the  $t \times t$  minors of the generic  $m \times n$  matrix  $(w_{i,j})$ . Write  $LT(I_t)$  for the ideal generated by the lead terms of the  $t \times t$  minors with respect to the graded reverse lexicographical order  $w_{1,1} > w_{1,2} > \cdots > w_{1,n} > w_{2,1} > \cdots > w_{2,n} > \cdots > w_{m,n}$ . Write  $\Delta_t$  for the Stanley-Reisner complex of  $LT(I_t)$ . Then the facets of  $\Delta_t$  correspond to all families of non-intersecting paths from  $w_{1,1}, w_{2,1}, \dots, w_{t-1,1}$  to  $w_{m,n}, w_{m,n-1}, \dots, w_{m,n-t+2}$ .*

Here, a path from  $w_{a,b}$  to  $w_{c,d}$ , given  $a \leq c$  and  $b \leq d$ , is a sequence of vertices starting at  $w_{a,b}$  and ending at  $w_{c,d}$  where each vertex in the sequence is either one step to the right or one step down from the previous vertex. A non-intersecting path of the kind described in the last line of the proposition above is a union of paths from  $w_{i,1}$  to  $w_{m,n-i+1}$  whose pairwise intersection is empty. (It is known that for the graded reverse lexicographic order as well,

the leading terms of the  $t \times t$  minors generate the ideal of leading terms of  $I_t$ .)

We observe that the monomials in  $A$  correspond to the generators of  $LT(I_2)$  and the monomials in  $E$  correspond to the generators of  $LT(I_3)$  (with the order specified in Proposition 2.1). So, for a facet  $F = F_x F_y$  of  $\Delta_{LT(J)}$ , we have that  $F_x$  is not in  $LT(I_2)$  and  $F_y$  is not in  $LT(I_3)$ . Therefore, by Proposition 2.1, we can state the following lemma:

**Lemma 2.2.**  *$F_x$  is a subset of a path from  $x_{1,1}$  to  $x_{m,n}$  and  $F_y$  is a subset of a pair of non-intersecting paths from  $y_{1,1}, y_{2,1}$  to  $y_{m,n}, y_{m,n-1}$*

We will continue by showing that for each facet  $F = F_x F_y$  of  $\Delta_{LT(J)}$ ,  $F_x$  is a non-empty string that contains at least two  $x$ -vertices. It is straightforward to see that  $x_{m,n} F$  can not be divisible by any of the generators of  $LT(J)$ . Hence, maximality of  $F$  implies that  $x_{m,n}$  is already in  $F$ . The next lemma shows that, in addition to  $x_{m,n}$ ,  $F$  must contain another  $x$ -vertex.

**Lemma 2.3.** *Let  $F$  be a facet of the simplicial complex  $\Delta_{LT(J)}$ . Then  $F$  must contain at least two  $x$ -vertices, one of which is  $x_{m,n}$ .*

*Proof.* We already know that  $F$  must have  $x_{m,n}$ . Suppose that it is the only  $x$ -vertex that  $F$  has. Consider then  $x_{m-1,n} F$ . We can easily check then that  $x_{m-1,n} F$  is not divisible by any of the monomials in  $A, B, C, D$ , or  $E$ . So,  $x_{m-1,n} F \in \Delta_{LT(J)}$  and maximality of  $F$  implies that  $x_{m-1,n}$  must already be in  $F$ , a contradiction to the assumption that the only  $x$ -vertex that  $F$  contains is  $x_{m,n}$ .  $\square$

*Notation:* Let  $F = F_x F_y$  be any facet and recall that, by Lemma 2.2,  $F_x$  is a subset of a path from  $x_{1,1}$  to  $x_{m,n}$ . Thus, for any two  $x$ -vertices in  $F_x$ , one is always to the north, west, or north-west of the other. Let  $\mu(F)$  denote the  $x$ -vertex that is furthest to north and furthest to the west of all other  $x$ -vertices in  $F_x$ . Thus,  $\mu(F) = x_{i,j}$  implies that  $i \leq c$  and  $j \leq d$  for all  $x_{c,d}$  in  $F_x$  (see Figure 3). Notice that  $\mu(F) \neq x_{m,n}$  (Lemma 2.3).

The next lemma deals with the  $F_y$  part of a facet  $F$  and, in particular, the lemma lists some of the  $y$ -vertices that must be present in a given facet.

**Lemma 2.4.** *Let  $F = F_x F_y$  be a facet of the simplicial complex  $\Delta_{LT(J)}$  with  $\mu(F) = x_{i,j}$ . Then  $F$  must contain  $y_{i,n}$  and  $y_{m,j}$ .*

*Proof.* To prove that  $F$  contains  $y_{i,n}$ , it suffices to show that  $y_{i,n} F$  is not divisible by a monomial in  $A, B, C, D$ , or  $E$ . Maximality of  $F$  would then imply that  $y_{i,n}$  must be in  $F$ .

Obviously,  $y_{i,n} F$  can not be divisible by a monomial in  $A$ . Also,  $y_{i,n} F$  can not be divisible by a monomial in  $B$  because otherwise it is easy to see that  $y_{i,n}$  would have to be to the south-east of  $\mu(F) = x_{i,j}$  - a contradiction. If  $y_{i,n} F$  were divisible by a monomial in  $D$ , then another straightforward verification shows that  $y_{i,n}$  must in a row below  $\mu(F) = x_{i,j}$ , which is impossible.

Suppose that  $y_{i,n}F$  is divisible by a monomial in  $C$ . Then, there must be some  $x_{c,d}$  and  $y_{s,t}$  in  $F$  such that  $x_{c,d}y_{s,t}y_{i,n}$  is in  $C$  (recall Figure 1). But then the only possible location of  $y_{s,t}$  is to the south-east of  $x_{i,j}$ . However,  $x_{i,j}y_{s,t}$  is in  $B$  and in  $F$  - a contradiction.

Finally, suppose that  $y_{i,n}F$  is divisible by a monomial in  $E$ . Then there must be some  $y_{a,b}$  and  $y_{c,d}$  in  $F$  such that  $y_{i,n}y_{a,b}y_{c,d}$  is in  $E$  (recall Figure 1). In particular, it must be the case that, say,  $y_{c,d}$  is to the south-west of  $y_{a,b}$  which, in turn, is to the south-west of  $y_{i,n}$ . But then either  $x_{i,j}y_{a,b}$  is in  $B$  or  $x_{i,j}y_{c,d}y_{a,b}$  is in  $D$  - a contradiction in both cases.

So,  $y_{i,n}F$  is not divisible by a monomial in  $A, B, C, D$ , or  $E$  which implies, as argued above, that  $F$  must contain  $y_{i,n}$ . We can similarly show that  $F$  must contain  $y_{m,j}$  as well.  $\square$

Now we are ready to describe the structure of all facets of  $\Delta_{LT(J)}$ . The following notation will be useful in the next theorem: for a given facet  $F = F_x F_y$  with  $\mu(F) = x_{i,j}$ , consider the following partition of the  $y$ -vertices based on the index  $(i, j)$ :  $R_1 = \{y_{s,t} \mid s \leq i, j < t\}$ ,  $R_2 = \{y_{s,t} \mid s \leq i, t \leq j\}$ ,  $R_3 = \{y_{s,t} \mid i < s, t \leq j\}$ ,  $R_4 = \{y_{s,t} \mid i < s, j < t\}$  (see Figure 2).

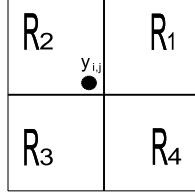


FIGURE 2.

**Theorem 2.5.** *Let  $F = F_x F_y$  be a facet of the simplicial complex  $\Delta_{LT(J)}$  with  $\mu(F) = x_{i,j}$ . Then  $F_x$  is a path from  $x_{i,j}$  to  $x_{m,n}$  and  $F_y$  is a family of non-intersecting paths from  $y_{1,1}, y_{2,1}$  to  $y_{i,n}, y_{m,j}$ .*

*Proof.* We will first show that  $F = F_x F_y$  as described in the theorem is indeed a valid facet of  $\Delta_{LT(J)}$ . Then we will argue that any facet of  $\Delta_{LT(J)}$  must have that form.

Let  $F = F_x F_y$  with  $F_x$  a path from  $x_{i,j}$  to  $x_{m,n}$  and  $F_y$  a family of non-intersecting paths from  $y_{1,1}, y_{2,1}$  to  $y_{i,n}, y_{m,j}$  be given (see Figure 3). We will first show that  $F$  is a facet of  $\Delta_{LT(J)}$ , i.e.  $F$  is not divisible by monomials in  $A, B, C, D$ , or  $E$  and  $F$  is maximal with respect to inclusion.

Obviously,  $F$  is not divisible by monomials in  $A$  and  $E$ . To see that  $F$  is not divisible by monomials in  $B, C$ , and  $D$ , it is enough to notice that  $F$  doesn't contain a  $y$ -variable in  $R_4$  or variables of the form  $y_{c,d}$  and  $y_{e,f}$  such that one is to the south-west of the other and both are entirely in  $R_1$  or  $R_3$ .

Next, we will show that  $F = F_x F_y$  is maximal with respect to inclusion by arguing that any vertex attached to  $F$  would make the resulting monomial

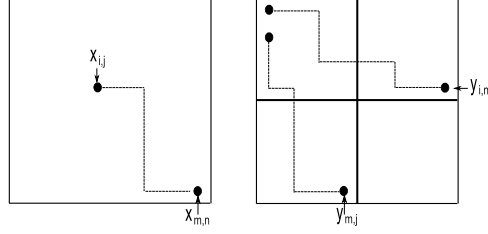


FIGURE 3.

divisible by some monomial in  $A$ ,  $B$ ,  $C$ ,  $D$ , or  $E$  (i.e. that resulting monomial can not be a face in  $\Delta_{LT(J)}$ ). Recall that by Lemma 2.2  $F_x$  is a subset of a path from  $x_{1,1}$  to  $x_{m,n}$ . So, if we attach a vertex  $x_{a,b}$  to  $F_x$ , it has to be to the north, west or north-west of  $x_{i,j}$ . But notice that in this case either  $x_{a,b}y_{m,j}$  or  $x_{a,b}y_{i,n}$ , or both, would be a monomial in  $B$  when  $i \neq m$ ,  $j \neq n$  (in the cases  $i = m$  or  $j = n$ ,  $x_{a,b}$  can also produce monomials in  $C$  and  $D$ ). So, no  $x$ -vertex can be attached to  $F$ . Recall also that  $F_y$  is a subset of a pair non-intersecting paths from  $y_{1,1}$ ,  $y_{2,1}$  to  $y_{m,n}$ ,  $y_{m,n-1}$  (Lemma 2.2). So, if we attach a vertex  $y_{c,d}$  to  $F_y$ , then  $y_{c,d}$  must be in one of those two non-intersecting paths. If  $i \neq m$ ,  $j \neq n$  (see Figure 4), then  $y_{c,d}$  must be in  $R_4$ , but then  $x_{i,j}y_{c,d}$  would be in  $B$ . If  $i = m$ , then  $y_{c,d}$  must be in  $R_1$  and in row  $m$ . But then  $x_{i,j}y_{c,d}$  and some  $y$ -variable that is in the upper path of  $F_y$  and in  $R_1$  would produce a monomial in  $C$ . Finally, if  $j = n$ , then  $y_{c,d}$  must be in  $R_3$  and in column  $n$ . But then  $x_{i,j}y_{c,d}$  and some  $y$ -variable that is in the lower path of  $F_y$  and in  $R_3$  would produce a monomial in  $D$ . So, no  $y$ -vertex can be attached to  $F$  either. Thus,  $F$  is maximal.

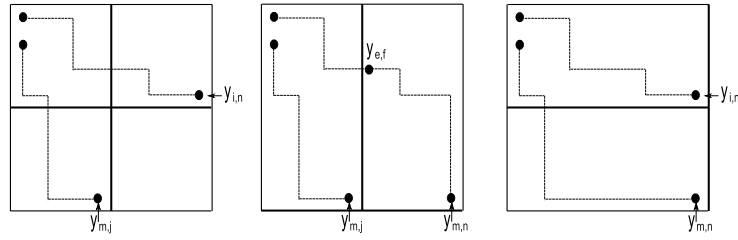


FIGURE 4.

Finally, we will show that any facet  $f = f_x f_y$  of  $\Delta_{LT(J)}$  with  $\mu(f) = x_{i,j}$  must be of the form described in the theorem. Since  $f_x$  is a subset of a path from  $x_{1,1}$  to  $x_{m,n}$  (by Lemma 2.2), and since  $\mu(f) = x_{i,j}$ , then it follows that  $f_x$  must actually be a subset of a path from  $x_{i,j}$  to  $x_{m,n}$ .

Next, again by Lemma 2.2,  $f_y$  must be a subset of a pair of non-intersecting paths from  $y_{1,1}$ ,  $y_{2,1}$  to  $y_{m,n}$ ,  $y_{m,n-1}$ . By Lemma 2.4, it follows that  $f_y$  must also contain  $y_{i,n}$  and  $y_{m,j}$ .

Now, if  $i \neq m$ ,  $j \neq n$  (see Figure 4),  $f_y$  can not contain  $y$ -vertices in  $R_4$ , because  $x_{i,j}$  and any vertex in that region is a monomial in  $B$ . So,  $f_y$  must be a subset of a family of two non-intersecting paths from  $y_{1,1}, y_{2,1}$  to  $y_{i,n}, y_{m,j}$ . Next, suppose  $i = m$  (see Figure 4). Since  $f_y$  is a subset of a pair of non-intersecting paths from  $y_{1,1}, y_{2,1}$  to  $y_{m,n}, y_{m,n-1}$ , it is straightforward to verify, using maximality of  $f$ , that  $f_y$  must contain the  $y$ -variable of the upper  $y$ -path that is furthest to north-west in  $R_1$ , call it  $y_{e,f}$ . Also, notice that there should be no  $y$ -vertices in  $f_y \cap R_1$  such that one is to the south-west of the other (otherwise  $x_{m,j}$  and those two  $y$ -vertices would produce a monomial in  $C$ ). Therefore,  $f_y \cap R_1$  must actually be a subset of a facet in  $\Delta_2$  on vertex set  $R_1$ , i.e.  $f_y \cap R_1$  must be a subset of some path in  $R_1$  starting in  $y_{e,f}$  and ending at  $y_{m,n}$  (recall Proposition 2.1). So,  $f_y$  must be a subset of a family of two non-intersecting paths from  $y_{1,1}, y_{2,1}$  to  $y_{i,n}, y_{m,j}$ ,  $i = m$ . Finally, we conclude the same result for the case  $j = n$  (see Figure 4) using similar arguments from the case  $i = m$ .

Finally, notice that  $f = f_x f_y$  is actually a subset of a some facet  $F$  as described in the theorem. Maximality of  $f$  implies that it actually has to be one of those facets  $F$ . □

Knowing the structure of a facet  $F = F_x F_y$  of the simplicial complex  $\Delta_{LT(J)}$ , we can easily count the number of vertices that  $F$  is composed of, so we can determine  $\dim F = |F| - 1$ . In particular, we see that the dimension of any facet  $F$  is  $2(m + n) - 3$ . Notice that the dimension of  $F$  depends only on the constants  $m$  and  $n$ . Thus, we can conclude that all facets of the simplicial complex  $\Delta_{LT(J)}$  have the same dimension, i.e.  $\Delta_{LT(J)}$  is a pure simplicial complex of dimension  $2(m + n) - 3$ .

**Corollary 2.6.** *The dimension of  $R/J$  is  $2(m + n) - 2$ .*

Theorem 2.5 also allows to determine the total number of facets in  $\Delta_{LT(J)}$ . Thus, we can determine the multiplicity of  $R/J$  as well.

**Corollary 2.7.** *The multiplicity of  $R/J$  is given by*

$$\sum_{(i,j), (i,j) \neq (m,n)} \binom{m+n-i-j}{m-i} \det \begin{pmatrix} \binom{i+n-2}{i-1} & \binom{m+j-2}{m-1} \\ \binom{i+n-3}{i-2} & \binom{m+j-3}{m-2} \end{pmatrix} \quad (1)$$

*Proof.* The number of paths from  $x_{i,j}$  to  $x_{m,n}$  is  $\binom{m+n-i-j}{m-i}$ , while the number of non-intersecting paths from  $y_{1,1}, y_{2,1}$  to  $y_{i,n}, y_{m,j}$  is given by (see [9, §2.2])

$$\det \begin{pmatrix} \binom{i+n-2}{i-1} & \binom{m+j-2}{m-1} \\ \binom{i+n-3}{i-2} & \binom{m+j-3}{m-2} \end{pmatrix}$$

□

*Remark 2.8.* Professor Sudhir Ghorpade ([2]) has shown that the expression for the multiplicity of  $R/J$  above simplifies to  $\binom{n+m-2}{m-1}^2$ .

### 3. SHELLABILITY OF $\Delta_{LT(J)}$

The main goal of this section is to prove that our simplicial complex  $\Delta_{LT(J)}$  is shellable. Recall the following definition of shellability:

*Definition 3.1.* A simplicial complex  $\Delta$  is *shellable* if it is pure and if its facets can be given a total order, say  $F_1, F_2, \dots, F_e$ , so that the following condition holds: for all  $i$  and  $j$  with  $1 \leq j < i \leq e$  there exists  $v \in F_i \setminus F_j$  and an index  $k$ ,  $1 \leq k < i$ , such that  $F_i \setminus F_k = \{v\}$ . A total order of the facets satisfying this condition is called *shelling* of  $\Delta$ .

**Theorem 3.2.** *The simplicial complex  $\Delta_{LT(J)}$  is shellable.*

*Proof.* Note that at the end of the previous section we have argued that  $\Delta_{LT(J)}$  is pure. We will proceed by first giving a partial order to the facets of  $\Delta_{LT(J)}$ . Let  $P = P_x P_y$  and  $Q = Q_x Q_y$  be two facets of  $\Delta_{LT(J)}$ . If  $\mu(P)$  is in a row below  $\mu(Q)$ , we set  $P < Q$  (see Figure 5). If  $\mu(P)$  and  $\mu(Q)$  are in the same row, but  $\mu(P)$  is to the right of  $\mu(Q)$ , we set  $P < Q$  (see Figure 5). If  $\mu(P) = \mu(Q)$  but  $P_x$  is to the right of  $Q_x$  as one goes from  $\mu(P)$  to  $x_{m,n}$ , then  $P < Q$  (see Figure 5). If  $P_x = Q_x$  and the upper  $y$ -path of  $P_y$  goes to the right of the upper  $y$ -path of  $Q_y$ , we set  $P < Q$ . Finally, if  $P_x = Q_x$ , the upper  $y$ -path of  $P_y$  is the same as the upper  $y$ -path of  $Q_y$  and the lower  $y$ -path of  $P_y$  goes to the right of the lower  $y$ -path of  $Q_y$ , we set  $P < Q$ . Now we arbitrarily extend this partial order on the facets of  $\Delta_{LT(J)}$  to a total order.

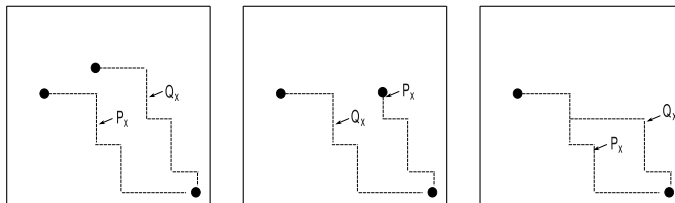


FIGURE 5.

Now we will prove that the selected total order is indeed a shelling of  $\Delta_{LT(J)}$ . Let  $P = P_x P_y$  and  $Q = Q_x Q_y$  be two facets of  $\Delta_{LT(J)}$  such that  $P < Q$ . Our goal is to find  $v \in Q \setminus P$  and a facet  $R < Q$  such that  $Q \setminus R = \{v\}$ . Suppose that  $\mu(P) \neq \mu(Q)$ . Notice  $P$  can not contain  $\mu(Q) = x_{i,j}$  (otherwise  $P < Q$  is contradicted). Take  $v = x_{i,j}$ . Take  $R = R_x R_y$  to be the following:  $R_x = Q_x \setminus x_{i,j}$  and  $R_y = Q_y y_{m,j+1}$  if  $\mu(R) = x_{i,j+1}$  or  $R_y = Q_y y_{i+1,n}$  if  $\mu(R) = x_{i+1,j}$ . In the special case  $\mu(Q) = x_{m-1,n}$ , take  $R_x = x_{m,n-1} x_{m,n}$ ,  $R_y = Q_y$ .



Next, suppose that  $\mu(P) = \mu(Q)$ , but  $P_x \neq Q_x$ . Then, there must be a right turn  $H = x_{a,b}$  in  $Q_x$  that is not in  $P_x$  or else  $Q_x$  would be to the right of  $P_x$ , contradicting  $P < Q$ . So, in this case take  $v = H = x_{a,b}$  and  $R = R_x R_y$  where  $R_x = Q_x$  with  $H = x_{a,b}$  replaced by  $x_{a+1,b-1}$  and  $R_y = Q_y$ .

Next, suppose that  $P_x = Q_x$  and the upper  $y$ -paths of the two facets are different. Notice that the upper path of  $Q_y$  can not be strictly on the right of the upper path of  $P_y$  (otherwise  $P < Q$  is contradicted). So, there must be a right turn  $H = y_{c,d}$  of the upper path of  $Q_y$  strictly on the left of the upper path of  $P_y$ . Thus,  $H = y_{c,d}$  can not be in  $P_y$ . So, take  $v = y_{c,d}$ . If  $y_{c+1,d-1}$  is not in the lower path of  $Q_y$ , let  $R = R_x R_y$  be the following facet:  $R_x = Q_x$  and  $R_y = Q_y$  with  $y_{c,d}$  replaced by  $y_{c+1,d-1}$ . If  $y_{c+1,d-1}$  is in the lower path of  $Q_y$  (see Figure 6), then notice that  $y_{c+1,d-1}$  must be a right turn as well. Then take  $R = R_x R_y$  to be the following facet:  $R_x = Q_x$  and  $R_y$  is obtained from  $Q_y$  by removing  $y_{c,d}$  and by adding  $y_{c+2,d-2}$ .

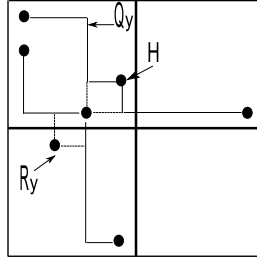


FIGURE 6.

Finally, suppose that  $P_x = Q_x$ , the upper  $y$ -paths of the two facets are the same, but the lower  $y$ -paths are different. Similarly as in the previous paragraph, we see that there must be a right turn  $H = y_{e,f}$  of the lower path of  $Q_y$  strictly on the left of the lower path of  $P_y$ . Notice that  $H = y_{e,f}$  can not be in the upper path of  $P_y$  because it is the same as the upper path of  $Q_y$ . So, take  $v = y_{e,f}$ . Let  $R = R_x R_y$  be the facet:  $R_x = Q_x$  and  $R_y = Q_y$  with  $y_{e,f}$  replaced by  $y_{e+1,f-1}$ .

□

We are now in position to prove Theorem 1.2, the main result of the paper:

*Proof of Theorem 1.2.* By standard results, the ring  $R/J$  is Cohen-Macaulay if the ring  $R/LT(J)$  is Cohen-Macaulay (see [6, Corollary 8.31]. By construction,  $R/LT(J)$  is precisely the Stanley-Reisner ring associated to  $\Delta_{LT(J)}$ , and since  $\Delta_{LT(J)}$  is shellable,  $R/LT(J)$  will necessarily be Cohen-Macaulay (see [1, Theorem 5.1.13]).

□

## REFERENCES

- [1] Bruns and Herzog, Cohen-Macaulay Rings, Cambridge Studies in Advanced Mathematics, 39, Cambridge University Press, 1993.
- [2] Sudhir Ghorpade, Private communication.
- [3] Sudhir Ghorpade, Young bitableaux, lattice paths and Hilbert functions, Journal of Statistical Planning and Inference, (Special issue on Lattice Path Combinatorics and Applications), **54**, 55–66, 1996.
- [4] Tomaž Košir and B.A. Sethuraman, Determinantal varieties over truncated polynomial rings, Journal of Pure and Applied Algebra, **195**, 75–95, 2005.
- [5] Tomaž Košir and B.A. Sethuraman, A Groebner basis for the  $2 \times 2$  determinantal ideal mod  $t^2$ , J. Algebra, **292**, 138–153, 2005.
- [6] Ezra Miller and Bernd Sturmfels, Combinatorial Commutative Algebra, Graduate Texts in Mathematics, Springer-Verlag, 2005.
- [7] Greg Smith and Jerzy Weyman, Resolutions of tangent schemes of determinantal ideals, preprint.
- [8] Winfried Bruns and Aldo Conca, Gröbner bases and determinantal ideals, Commutative algebra, singularities and computer algebra (Sinaia, 2002), 9–66, NATO Sci. Ser. II Math. Phys. Chem., 115, Kluwer Acad. Publ., Dordrecht, 2003.
- [9] C. Krattenthaler, The major counting of nonintersecting lattice paths and generating functions for tableaux, Mem. Amer. Math. Soc., **115**, number 552, Providence, RI, 1995.

DEPT. OF MATHEMATICS, CALIFORNIA STATE UNIVERSITY NORTHRIDGE, NORTHRIDGE  
 CA 91330, U.S.A., CURRENT ADDRESS: DEPT OF MATHEMATICS, UNIVERSITY OF CAL-  
 IFORNIA SANTA BARBARA, SANTA BARBARA CA 93106  
*E-mail address:* boyan@math.ucsb.edu