An example of constrained evolution:
6-3=5
June 13, 2006 A.Alekseenko

1. Six wave equations coupled through three constraint equations

Let κ_{ij} be a symmetric matrix field in Ω, consider the system of equations

$$\partial_t^2 \kappa_{ij} = \partial^l \partial_l \kappa_{ij} \tag{1}$$

coupled through the constraint

$$M_j = \partial^i \kappa_{ij} - \partial_j \kappa^i_i = 0. \tag{2}$$

System (1), (2) is a simplified model problem derived from the BSSN system [?? ??] in general relativity under the linearization assumption (with unit lapse and zero shift) by taking a second order reduction in time (see Alekseeenko [??] for detail). Here, κ_{ij} plays the role of perturbation of the extrinsic curvature and M_i is the linearized momentum constraint.

Lemma 1. *If $\kappa_{ij}(x,t)$ is a solution of (1) then the constraint quantity M_j defined by (2) obeys the vector wave equation:*

$$\partial_t^2 M_i = \partial^l \partial_l M_i \tag{3}$$

Proof. The proof follows by substitution, commuting derivatives, and using (1). □

Similarly to the previous section, we want to construct boundary data for (1) that is consistent with (2). There are examples of such data known, in particular, the homogeneous constraint preserving Dirichlet data proposed in [??], and the inhomogeneous Dirichlet data for the first order reduction of (1) [?? ?? ??], obtained by integration along the boundary. However, examples of the radiative data are limited, and those existing are lacking rigor. We eliminate this gap by using the trivial constraint evolution reduction method. In particular, we prove that problem (1), (2) admits well-posed (maximally dissipative) boundary conditions with five modes freely specifiable. On the basis of these conditions, constraint preserving boundary conditions for the free evolution problem are constructed.

1.1. The trivial constraint evolution reduction.

The auxiliary problem consists of the equation

$$\partial_t^2 \kappa_{ij} = \partial^l \partial_l \kappa_{ij} - 2\partial_{(i} M_{j)} + \frac{1}{2}\delta_{ij}\partial^l M_l \tag{4}$$

subject to the constraint (2).

Lemma 2. *Evolution of constraint quantity in system (4), (2) is trivial. That is, let κ_{ij} be any (sufficiently smooth) solution to (4), then*

$$\partial_t^2 M_i \equiv 0. \tag{5}$$

Proof. The proof follows from using the definition of M_i (equation (2)), commuting derivatives, and substituting (4) for the time derivatives of κ_{ij}. □

We now proceed with the derivation. First of all, substituting the definition of M_i into (4) we write it explicitly in terms of κ_{ij} as

$$\partial_t^2 k_{ij} = \partial^l \partial_l k_{ij} - 2\partial_{(i}\partial^l \kappa_{|l|j)} + 2\partial_i\partial_j k + \frac{1}{2}\delta_{ij}\partial^l\partial^m \kappa_{lm} - \frac{1}{2}\delta_{ij}\partial^l\partial_l \kappa^m_m. \tag{6}$$

Let us denote by $\mathbb{T}$ the space of all triple-indexed arrays $\omega_{pqr} \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$ which are symmetric in the last two indices, $\omega_{pqr} = \omega_{prq}$. Notice that at each point, the gradient $\partial_p \kappa_{qr} \in \mathbb{T}$. We introduce the linear algebraic operator $L_{lij}^{pqr} : \mathbb{T} \to \mathbb{T}$ by the formula

$$L_{lij}^{pqr} = \delta_l^p \delta_{(i}^q \delta_{j)}^r - \delta_{(i}^p \delta_{|l|}^q \delta_{j)}^r - \delta_{l(i}\delta_{j)}^r \delta^{pq} + 2\delta_{l(i}\delta_{j)}^p \delta^{qr} + \frac{1}{2}\delta_{ij}\delta^{pq}\delta_l^r - \frac{1}{2}\delta_{ij}\delta_l^p \delta^{qr}.$$

In terms of the operator L_{lij}^{pqr} which we call *the algebraic operator of the second order equation*, (6) can be restated in the divergence form

$$\partial_t^2 k_{ij} = \partial^l L_{lij}^{pqr} \partial_p k_{qr}. \tag{7}$$

Lemma 3. *Let vectors n_i, m_i, and l_i constitute an orthonormal triple in $\mathbb{R}^3$, the (right) eigenvalues and eigenvectors of L_{lij}^{pqr}, that is solutions μ, ω_{pqr} to the equation $L_{lij}^{pqr}\omega_{pqr} = \alpha\omega_{lij}$, are given by*

$$\begin{aligned}
&\alpha = \frac{3}{2}: \\
&\quad \omega 1_{pqr} = \frac{1}{\sqrt{3}}[2n_p m_{(q}l_{r)} - m_p l_{(q}n_{r)} - l_p n_{(q}m_{r)}] \\
&\quad \omega 2_{pqr} = m_p l_{(q}n_{r)} - l_p n_{(q}m_{r)} \\
&\quad \omega 3_{pqr} = \frac{1}{\sqrt{3}}[n_p(m_q m_r - l_q l_r) - (m_p m_{(q} - l_p l_{(q})n_{r)}] \\
&\quad \omega 4_{pqr} = \frac{1}{\sqrt{3}}[m_p(l_q l_r - n_q n_r) - (l_p l_{(q} - n_p n_{(q})m_{r)}] \\
&\quad \omega 5_{pqr} = \frac{1}{\sqrt{3}}[l_p(n_q n_r - m_q m_r) - (n_p n_{(q} - m_p m_{(q})l_{r)}] \\
&\alpha = 0: \\
&\quad \omega 6_{pqr} = n_p m_{(q}l_{r)} + m_p l_{(q}n_{r)} + l_p n_{(q}m_{r)} \\
&\quad \omega 7_{pqr} = n_p(m_q m_r - l_q l_r) + 2(m_p m_{(q} - l_p l_{(q})n_{r)} \\
&\quad \omega 8_{pqr} = m_p(l_q l_r - n_q n_r) + 2(l_p l_{(q} - n_p n_{(q})m_{r)} \\
&\quad \omega 9_{pqr} = l_p(n_q n_r - m_q m_r) + 2(n_p n_{(q} - m_p m_{(q})l_{r)}
\end{aligned}$$

$$\omega 10_{pqr} = n_p n_q n_r - \frac{1}{2} n_p (m_q m_r + l_q l_r) - (m_p m_{(q} + l_p l_{(q}) n_{r)}$$
$$\omega 11_{pqr} = m_p m_q m_r - \frac{1}{2} m_p (l_q l_r + n_q n_r) - (l_p l_{(q} + n_p n_{(q}) m_{r)}$$
$$\omega 12_{pqr} = l_p l_q l_r - \frac{1}{2} l_p (n_q n_r + m_q m_r) - (n_p n_{(q} + m_p m_{(q}) l_{r)}$$

$$\alpha = \frac{1}{2} :$$
$$\omega 13_{pqr} = \frac{1}{\sqrt{2}} [\delta_{p(q} n_{r)}]$$
$$\omega 14_{pqr} = \frac{1}{\sqrt{2}} [\delta_{p(q} m_{r)}]$$
$$\omega 15_{pqr} = \frac{1}{\sqrt{2}} [\delta_{p(q} l_{r)}]$$

$$\alpha = 0 :$$
$$\omega 16_{pqr} = \frac{1}{\sqrt{115}} [8\delta_{p(q} n_{r)} - n_p \delta_{qr}]$$
$$\omega 17_{pqr} = \frac{1}{\sqrt{115}} [8\delta_{p(q} m_{r)} - m_p \delta_{qr}]$$
$$\omega 18_{pqr} = \frac{1}{\sqrt{115}} [8\delta_{p(q} l_{r)} - l_p \delta_{qr}]$$

Proof. For the proof, recall that a symmetric matrix κ_{qr} is spanned by $\{n_q n_r, m_q m_r + l_q l_r, m_q m_r - l_q l_r, n_{(q} m_{r)}, n_{(q} l_{r)}, m_{(q} l_{r)}\}$. Correspondingly, $\partial_p \kappa_{qr}$ is spanned by $\{n_q, m_q, l_q\} \times \{n_q n_r, m_q m_r + l_q l_r, m_q m_r - l_q l_r, n_{(q} m_{r)}, n_{(q} l_{r)}, m_{(q} l_{r)}\}$. Then the lemma follows from the direct substitution. □

Remark. Notice that the eigenvectors $\omega 1_{pqr}, \ldots, \omega 12_{pqr}$ are mutually perpendicular in the sense of contraction with the flat metric: $\langle \omega_{pqr}, \eta_{pqr} \rangle = \omega_{pqr} \eta^{pqr}$, while the rest of the eigenvectors form the three pairs: $\{\omega 13_{pqr}, \omega 16_{pqr}\}$, $\{\omega 14_{pqr}, \omega 17_{pqr}\}$, and $\{\omega 15_{pqr}, \omega 18_{pqr}\}$, which are not perpendicular within the pair, but orthogonal to the rest of the eigenvectors. Failure of orthogonality among the eigenvectors means that L^{pqr}_{lij} is not symmetric. However, a symmetric positive definite linear operator B^{pqr}_{lij} can be easily constructed using the left eigenvectors of L^{pqr}_{lij} such that all eigenvectors are mutually perpendicular in the scalar product $\langle \lambda_{pqr}, \eta_{pqr} \rangle_B = \lambda^{lij} B^{pqr}_{lij} \eta^{pqr}$, so L^{pqr}_{lij} is symmetrizable.

Remark. The fact that L^{pqr}_{lij} is not symmetric implies that there is no straightforward first order symmetric hyperbolic reduction of (6). For example, if one introduces λ_{lij} and η_{lij} as the new variables, the characteristic variables of the resulting first order system have zero and imaginary eigenspeeds only as the direct consequence of non-orthogonality of eigenfields η_{lij} and ρ_{lij}. A symmetric first order reduction of (6) therefore relies on existence of a positive definite algebraic operator N^{ij}_{st} with a property: $N^{ij}_{st} \partial^l L^{pqr}_{lij} = \partial^l B^{kij}_{lst} L^{pqr}_{kij}$. Construction of such N^{ij}_{st}, however is expected to be difficult (if possible at all), and not needed for the well-posedness result below.

1.2. **System's natural energy and the boundary conditions.** The spectral structure of L^{pqr}_{lij} suggests to split gradient of κ_{ij} into the four components:

$$\partial_p \kappa_{qr} = \lambda_{pqr} + \mu_{pqr} + \eta_{pqr} + \rho_{pqr}, \tag{8}$$

where (denoting $\kappa = \kappa^i_i$)

$$\begin{aligned}
\lambda_{pqr} &= \frac{2}{3}[\partial_p \kappa_{qr} - \partial_{(q}\kappa_{r)p} - \frac{1}{2}\delta_{p(q}\partial^s \kappa_{|s|r)} + \frac{1}{2}\delta_{p(q}\partial_{r)}\kappa + \frac{1}{2}\delta_{qr}\partial^s\kappa_{sp} - \frac{1}{2}\delta_{qr}\partial_p\kappa],\\
\mu_{pqr} &= \frac{1}{3}[\partial_p \kappa_{qr} + 2\partial_{(q}\kappa_{r)p} - \frac{4}{5}\delta_{p(q}\partial^s \kappa_{|s|r)} - \frac{2}{5}\delta_{p(q}\partial_{r)}\kappa - \frac{2}{5}\delta_{qr}\partial^s\kappa_{sp} - \frac{1}{5}\delta_{qr}\partial_p\kappa],\\
\eta_{pqr} &= -[\delta_{p(q}\partial^s\kappa_{|s|r)}] + 3[\delta_{p(q}\partial_{r)}\kappa],\\
\rho_{pqr} &= \frac{1}{5}[8\delta_{p(q}\partial^s\kappa_{|s|r)} - \delta_{qr}\partial^s\kappa_{sp}] - \frac{2}{5}[8\delta_{p(q}\partial_{r)}\kappa - \delta_{qr}\partial_p\kappa].
\end{aligned}$$

The next lemma summarizes properties of decomposition (8).

Lemma 4. *Let fields λ_{pqr}, μ_{pqr} η_{pqr}, ρ_{pqr} be defined by (8), then*

(a) *at each point, λ_{pqr} is spanned by $\omega 1_{pqr}, \dots, \omega 5_{pqr}$, μ_{pqr} by $\omega 6_{pqr}, \dots, \omega 12_{pqr}$, η_{pqr} by $\omega 13_{pqr}, \omega 14_{pqr}, \omega 15_{pqr}$, and ρ_{pqr} by $\omega 16_{pqr}, \omega 17_{pqr}, \omega 18_{pqr}$ correspondingly;*

(b) *at each point, fields λ_{pqr}, μ_{pqr} η_{pqr}, ρ_{pqr} are eigenvectors of L^{pqr}_{lij} with eigenvalues, respectively, $3/2$, 0, $1/2$, and 0;*

(c) *fields λ_{pqr} satisfy the cyclic identity $\lambda_{pqr}+\lambda_{qrp}+\lambda_{rpq} = 0$, in addition, λ_{pqr} is trace free with respect to second and third indices, that is $\delta^{qr}\lambda_{pqr} = 0$. From these two properties one can deduce that λ_{pqr} is also trace free in first and third index, that is $\delta^{pr}\lambda_{pqr} = 0$. Fields μ_{pqr} are trace free in both pairs of indices pq and qr.*

(d) *the following table summarizes the mutual orthogonality of λ_{pqr}, μ_{pqr} η_{pqr}, ρ_{pqr} with respect the to usual scalar product, $\langle \nu_{pqr}, \sigma_{pqr}\rangle = \nu_{pqr}\sigma^{pqr}$:*

	λ	μ	η	ρ
λ	$\parallel$	$\perp$	$\perp$	$\perp$
μ	$\perp$	$\parallel$	$\perp$	$\perp$
η	$\perp$	$\perp$	$\parallel$	$\not\perp$
ρ	$\perp$	$\perp$	$\not\perp$	$\parallel$

Proof. (a) To prove that λ_{pqr} is spanned by $\omega 1_{prq}, \dots \omega 5_{pqr}$ it is enough to check that it is orthogonal (and thus linearly independent) to the rest of the eigenvectors $\omega 6_{prq}, \dots \omega 18_{pqr}$. Similarly, μ_{pqr} is orthogonal to $\omega 1_{prq}, \dots \omega 5_{pqr}, \omega 13_{prq}, \dots \omega 18_{pqr}$, thus is spanned by $\omega 6_{prq}, \dots \omega 12_{pqr}$. Notice that η_{pqr} and ρ_{pqr} are not mutually perpendicular, but both are orthogonal to $\omega 1_{prq}, \dots \omega 12_{pqr}$. At the same time, η_{pqr} and ρ_{pqr} decompose into a linear combination of $\omega 13_{pqr}, \omega 14_{pqr}, \omega 15_{pqr}$ and $\omega 16_{pqr}, \omega 17_{pqr}, \omega 18_{pqr}$ with obvious coefficients.

Part (b) follows directly from (a). Let us prove (c). To prove the cyclic identity, it is enough to show that $\lambda_{pqr} = -\lambda_{qrp} - \lambda_{rqp}$. Indeed,

$$\begin{aligned}\lambda_{pqr} = \frac{2}{3}[\partial_p\kappa_{qr} - \frac{1}{2}\partial_q\kappa_{rp} - \frac{1}{2}\partial_r\kappa_{qp} - \frac{1}{4}\delta_{pq}\partial^s\kappa_{sr} - \frac{1}{4}\delta_{pr}\partial^s\kappa_{sq}\\ + \frac{1}{4}\delta_{pq}\partial_r\kappa + \frac{1}{4}\delta_{pr}\partial_q\kappa + \frac{1}{2}\delta_{qr}\partial^s\kappa_{sp} - \frac{1}{2}\delta_{qr}\partial_p\kappa]\\ = \frac{2}{3}[-\partial_q\kappa_{rp} + \frac{1}{2}\partial_p\kappa_{qr} + \frac{1}{2}\partial_r\kappa_{qp} + \frac{1}{4}\delta_{rq}\partial^s\kappa_{sp} + \frac{1}{4}\delta_{pq}\partial^s\kappa_{sr}\\ - \frac{1}{4}\delta_{qr}\partial_p\kappa - \frac{1}{4}\delta_{qp}\partial_r\kappa - \frac{1}{2}\delta_{pr}\partial^s\kappa_{sq} + \frac{1}{2}\delta_{rp}\partial_q\kappa]\\ + \frac{2}{3}[-\partial_r\kappa_{qp} + \frac{1}{2}\partial_p\kappa_{qr} + \frac{1}{2}\partial_q\kappa_{rp} + \frac{1}{4}\delta_{rq}\partial^s\kappa_{sp} + \frac{1}{4}\delta_{rp}\partial^s\kappa_{sq}\\ - \frac{1}{4}\delta_{rq}\partial_p\kappa - \frac{1}{4}\delta_{rp}\partial_q\kappa - \frac{1}{2}\delta_{pq}\partial^s\kappa_{sr} + \frac{1}{2}\delta_{qp}\partial_r\kappa]\\ = -\lambda_{qrp} - \lambda_{rqp}.\end{aligned}$$

The trace free property is verified by the direct substitution:

$$\begin{aligned}\delta^{qr}\lambda_{pqr} &= \frac{2}{3}\delta^{qr}[\partial_p\kappa_{qr} - \partial_{(q}\kappa_{r)p} - \frac{1}{2}\delta_{p(q}\partial^s\kappa_{|s|r)} + \frac{1}{2}\delta_{p(q}\partial_{r)}\kappa + \frac{1}{2}\delta_{qr}\partial^s\kappa_{sp} - \frac{1}{2}\delta_{qr}\partial_p\kappa]\\ &= \frac{2}{3}[\partial_p\kappa - \partial^q\kappa_{qp} - \frac{1}{2}\partial^s\kappa_{sp} + \frac{1}{2}\partial_p\kappa + \frac{3}{2}\partial^s\kappa_{sp} - \frac{3}{2}\partial_p\kappa] = 0,\end{aligned}$$

By contracting the cyclic identity and using the trace free property $\delta^{qr}\lambda_{pqr} = 0$,

$$0 = \delta^{qr}(\lambda_{pqr} + \lambda_{qrp} + \lambda_{rpq}) = 2\delta^{qr}\lambda_{qrp}.$$

Similarly, one can check by direct substitution that $\delta^{pq}\mu_{pqr} = 0$, and $\delta^{qr}\mu_{pqr} = 0$.

Part (d) essentially summarizes proof of part (a), but one can quickly see that the trace free properties of λ_{pqr} and μ_{pqr} imply that their orthogonality to η_{pqr} and ρ_{pqr}. The orthogonality between λ_{pqr} and μ_{pqr} can be verified by using the trace free properties and then the cyclic property of λ_{pqr}. □

Remark. Because of the trace free and cyclic properties, which are purely algebraic properties of the corresponding fields, the orthogonality between the fields remains even if one replaces κ_{ij} with different matrix fields in definitions of λ, μ, η, and ρ. Thus, decomposition (8) should be thought rather as a projection on different subspaces of the gradient. This important observation will simplify our calculations essentially when deriving energy estimates.

Substituting decomposition (8) and using Lemma 4 part (b), we rewrite (7) as

$$\partial_t^2 k_{ij} = \partial^l(\frac{3}{2}\lambda_{lij} + \frac{1}{2}\eta_{lij}). \tag{9}$$

Remark. Note that (9) is as a second order equation on κ_{ij}, since λ_{lij} and η_{lij} may be replaced with their definitions at any moment. However, we will keep λ_{lij} and η_{lij} for some time as shorthand notations for the corresponding parts of the gradient of κ_{ij}: first, because we want to use the structure and the spanning properties of these fields to simplify proofs; second, because the boundary conditions that we are about to write look rather confusing in terms of κ_{ij}, but clearly make sense as the normal components of fields of λ_{lij} and η_{lij}.

To formulate the boundary conditions, we need to introduce some more notations: let vectors n_i, m_i, and l_i constitute an orthonormal triple in $\mathbb{R}^3$, the following vectors form the basis in the space of all symmetric matrices $\mathbb{S}$ (MOVE THIS DEFINITION UP):

$$e1_{ij} = \sqrt{2}\, n_{(i}m_{j)}, \quad e2_{ij} = \sqrt{2}\, n_{(i}l_{j)}, \quad e3_{ij} = \sqrt{2}\, m_{(i}l_{j)},$$
$$e4_{ij} = \frac{1}{\sqrt{2}}(m_i m_j - l_i l_j), \quad e5_{ij} = n_i n_j, \quad e6_{ij} = \frac{1}{\sqrt{2}}(m_i m_j + l_i l_j).$$

Let each of the constants $a1, \ldots, a5$ be either 0 or 1, we introduce the projection operator $(P_5)_{ij}^{pq} : \mathbb{S} \to \operatorname{span}\{e1, e2, e3, e4, e5\} \subset \mathbb{S}$ by the formula

$$(P_5)_{ij}^{pq} = a1\, e1^{pq} e1_{ij} + \cdots + a5\, e5^{pq} e5_{ij}.$$

Let n_i be the outward unit normal to the boundary $\partial\Omega$, the following (homogeneous) condition will be imposed on fields κ_{ij} at the boundary,

$$(P_5)_{ij}^{pq} \partial_t \kappa_{pq} + 3n^p \lambda_{pij} + n^p \eta_{pij} = 0, \qquad \text{on } \partial\Omega. \tag{10}$$

Here we assume that the vector of coefficients $a = (a1, \ldots, a5)$ consists of 0s and 1s which may or may not change within the face. The value 0 corresponds to the Neumann type data on the corresponding components, while 1 gives the radiative type data. Notice that we intentionally skip the case of Dirichlet data, mostly because the examples of Dirichlet data (and especially, the homogeneous) for systems arising in numerical relativity are well known in the literature [?? ?? ???], but also for simplicity of presentation. However, one can easily see that generalization of proofs to include case of the Dirichlet data is straightforward.

Remark. Generalization to the inhomogenous boundary data is, however, more involved. First of all, because the equations are second order both in space and in time, the energy argument below does not extend to inhomogeneous boundary data. Instead, one may try reduce the problem to the case of homogeneous boundary conditions. In particular, the reductions formulated in [Kreiss and Lorentz, Chapter 7]) can be applied to our problem in the case when coefficients of (10) are constant along the face (and both boundary and initial data are consistent).

Theorem 5. *Let κ_{ij} be a (sufficiently smooth) solution to (6) ((7), or (9)) satisfying boundary condition (10) (for either pair (a, b)). Then the solution κ_{ij} is unique and on the time interval $0 \le t \le T$ satisfies the energy estimate*

$$\begin{aligned} &\|\partial_t k_{ij}\|^2 + \frac{3}{2}\|\lambda_{lij}\|^2 + \|\partial^s \kappa_{sj} - 3M_j\|^2 + 9\|M_j\|^2 \\ &\quad \le \mathrm{e}^{3T/2}[\|\partial_t k_{ij}(0)\|^2 + \frac{3}{2}\|\lambda_{lij}(0)\|^2 + \|\partial^s \kappa_{sj}(0) - 3M_j(0)\|^2 + 9\|M_j\|^2(0)] \\ &\quad + \int_0^T F(t)\, \mathrm{e}^{3(T-t)/2} dt. \end{aligned} \tag{11}$$

where $F(t) = \frac{45}{4}\|M_j(0)\|^2 + (\frac{51}{4} + \frac{45}{2}t)\|\partial_t M_j(0)\|^2$.

Proof. First of all, let us verify that condition (10) makes sense, namely, that the quantity $(3n^p \lambda_{pij} + n^p \eta_{pij})$ is spanned by $\{e1, \ldots, e5\}$. Choose a face of the boundary, let n_i be the outward unit normal, and vectors l_i and m_i be mutually perpendicular unit vectors tangential to boundary. According to Lemma 4, in a small neighborhood of the face, λ_{pij} is spanned by $\omega1_{pij}, \ldots, \omega5_{pij}$. In other words,

$$\lambda_{pij} = \lambda 1 \omega 1_{pij} + \ldots + \lambda 5 \omega 5_{pij},$$

where $\lambda 1 = \lambda_{pij}\omega 1^{pij}$, ..., $\lambda 5 = \lambda_{pij}\omega 5^{pij}$. Also,

$$\eta_{pij} = \eta 1\omega 13_{pij} + \eta 2\omega 14_{pij} + \eta 3\omega 15_{pij},$$

where $\eta 1 = \sqrt{2}(-\partial^l\kappa_{lj} + 3\partial_j\kappa)n^j$, $\eta 2 = \sqrt{2}(-\partial^l\kappa_{lj} + 3\partial_j\kappa)m^j$, $\eta 3 = \sqrt{2}(-\partial^l\kappa_{lj} + 3\partial_j\kappa)l^j$. Substituting expressions for λ_{pij}, and η_{pij} into $(3n^p\lambda_{pij}+n^p\eta_{pij})$ we obtain after simplification

$$(\sqrt{\frac{3}{2}}\lambda 4 + \frac{1}{2}\eta 2)e1_{ij} + (-\sqrt{\frac{3}{2}}\lambda 5 + \frac{1}{2}\eta 3)e2_{ij} + \sqrt{6}\lambda 1\, e3_{ij} + \sqrt{6}\lambda 3\, e4_{ij} + \frac{1}{\sqrt{2}}\eta 1\, e5_{ij}.$$

Thus the expanded version of (10) is (here $\kappa 1 = \kappa_{ij}e1^{ij}$, ..., $\kappa 5 = \kappa_{ij}e5^{ij}$, and similarly, $g1 = g_{ij}e1^{ij}$, ..., $g5 = g_{ij}e5^{ij}$):

$$a1\partial_t\kappa 1 + \sqrt{\frac{3}{2}}\lambda 4 + \frac{1}{2}\eta 2 = 0, \quad a2\partial_t\kappa 2 - \sqrt{\frac{3}{2}}\lambda 5 + \frac{1}{2}\eta 3 = 0,$$
$$a3\partial_t\kappa 3 + \sqrt{6}\lambda 1 = 0, \quad a4\partial_t\kappa 4 + \sqrt{6}\lambda 3 = 0, \quad a5\partial_t\kappa 5 + \frac{1}{\sqrt{2}}\eta 1 = 0.$$

Now we will establish the estimate (11). By contracting equation (9) with $\partial_t\kappa_{ij}$ from both sides and integrating over Ω, we get

$$\int_\Omega \partial_t^2 k_{ij}(\partial_t\kappa^{ij})dx = \int_\Omega \partial^l(\frac{3}{2}\lambda_{lij} + \frac{1}{2}\eta_{lij})(\partial_t\kappa^{ij})dx.$$

By rearranging the terms in the left side and integrating by parts in the right side we get

$$\frac{1}{2}\partial_t\|\partial_t k_{ij}\|^2 = -\int_\Omega(\frac{3}{2}\lambda_{lij} + \frac{1}{2}\eta_{lij})(\partial_t\partial^l\kappa^{ij})dx + \int_{\partial\Omega} n^p(\frac{3}{2}\lambda_{pij} + \frac{1}{2}\eta_{pij})(\partial_t\kappa^{ij})d\sigma.$$

Consider the volume integral in the last expression. Using decomposition (8) we rewrite it as

$$\int_\Omega(\frac{3}{2}\lambda_{lij} + \frac{1}{2}\eta_{lij})(\partial_t(\lambda^{lij} + \mu^{lij} + \eta^{lij} + \rho^{lij}))dx.$$

Using orthogonality between the fields (see the remark to Lemma 4), the latter simplifies into

$$\int_\Omega[\frac{3}{2}\lambda_{lij}(\partial_t\lambda^{lij}) + \frac{1}{2}\eta_{lij}(\partial_t(\eta^{lij} + \rho^{lij})]dx.$$

Substituting definitions of η_{lij} and ρ_{lij} we simplify the second term to

$$\int_\Omega[\frac{3}{2}\lambda_{lij}(\partial_t\lambda^{lij}) + \frac{1}{2}(-\partial^s\kappa_{sj} + 3\partial_j\kappa)(\partial_t\partial^p\kappa_p^j)]dx.$$

Notice that the first term in this integral is in the convenient “energy” form, so we rewrite the expression,

$$\frac{1}{2}\partial_t\frac{3}{2}\|\lambda_{lij}\|^2 + \frac{1}{2}\int_\Omega(-\partial^s\kappa_{sj} + 3\partial_j\kappa)(\partial_t\partial^p\kappa_p^j)dx.$$

The second term can also be transformed to the energy form. To do so, first, we rewrite $-\partial^s\kappa_{sj} + 3\partial_j\kappa = 2\partial^s\kappa_{sj} - 3M_j$, then

$$\begin{aligned}
&\int_\Omega (\partial^s\kappa_{sj} - \frac{3}{2}M_j)(\partial_t\partial^p\kappa_p^j)dx \\
&\qquad = \frac{1}{2}\partial_t\|\partial^s\kappa_{sj} - 3M_j\|^2 + \int_\Omega (\partial^s\kappa_{sj} - \frac{3}{2}M_j)\partial_t(\frac{3}{2}M_j)dx \\
&\qquad = \frac{1}{2}\partial_t(\|\partial^s\kappa_{sj} - 3M_j\|^2 - \frac{9}{4}\|M_j\|^2) + \frac{3}{2}\int_\Omega (\partial^s\kappa_{sj})\partial_t M_j dx.
\end{aligned}$$

Finally, we write our basic integral identity:

$$\begin{aligned}
&\partial_t[\|\partial_t k_{ij}\|^2 + \frac{3}{2}\|\lambda_{lij}\|^2 + \|\partial^s\kappa_{sj} - 3M_j\|^2 - \frac{9}{4}\|M_j\|^2] \\
&\qquad = 3\int_\Omega (\partial^s\kappa_{sj})\partial_t M_j dx + \int_{\partial\Omega} n^p(3\lambda_{pij} + \eta_{pij})(\partial_t\kappa^{ij})d\sigma.
\end{aligned} \tag{12}$$

The next step in the proof is standard: the boundary condition (10) implies that the boundary integral is non-positive and the volume integral can be estimated using the elementary inequality $2u_iv_i \le u_iu^i + v_iv^i$,

$$\begin{aligned}
&\partial_t[\|\partial_t k_{ij}\|^2 + \frac{3}{2}\|\lambda_{lij}\|^2 + \|\partial^s\kappa_{sj} - 3M_j\|^2 - \frac{9}{4}\|M_j\|^2] \\
&\qquad \le \frac{3}{2}[\|\partial^s\kappa_{sj}\|^2 + \|\partial_t M_j\|^2].
\end{aligned}$$

Notice that the trivial evolution of the constraint quantity plays an important role in the energy estimate. Indeed, since M_j evolves trivially, for any solution of (6) it is mandatory that $\partial^s\kappa_{sj} - \partial_j\kappa = M_j = M_j(0) + t(\partial_t M_i)(0)$. In other words, one can treat $\|\partial_t M_j\|$ and $\partial_t\|M_j\|^2$ as known functions. With this, the rest of the estimate is straightforward. By adding $\partial_t\frac{45}{4}\|M_j\|^2$ to both sides of inequality, we get

$$\begin{aligned}
&\partial_t[\|\partial_t k_{ij}\|^2 + \frac{3}{2}\|\lambda_{lij}\|^2 + \|\partial^s\kappa_{sj} - 3M_j\|^2 + 9\|M_j\|^2] \\
&\qquad \le \frac{3}{2}[\|\partial^s\kappa_{sj}\|^2 + \|\partial_t M_j\|^2] + \frac{45}{4}\partial_t\|M_j\|^2.
\end{aligned}$$

Then, by the triangle inequality,

$$\begin{aligned}
&\partial_t[\|\partial_t k_{ij}\|^2 + \frac{3}{2}\|\lambda_{lij}\|^2 + \|\partial^s\kappa_{sj} - 3M_j\|^2 + 9\|M_j\|^2] \\
&\qquad \le \frac{3}{2}[\|\partial^s\kappa_{sj} - 3M_j\|^2 + 9\|M_j\|^2 + \|\partial_t M_j\|^2] + \frac{45}{4}\partial_t\|M_j\|^2.
\end{aligned}$$

Finally,

$$\partial_t\epsilon \le \frac{3}{2}\epsilon + F(t),$$

where $\epsilon = \|\partial_t k_{ij}\|^2 + \frac{3}{2}\|\lambda_{lij}\|^2 + \|\partial^s\kappa_{sj} - 3M_j\|^2 + 9\|M_j\|^2$, and $F(t) = \frac{45}{4}\|M_j(0)\|^2 + (\frac{51}{4} + \frac{45}{2}t)\|\partial_t M_j(0)\|^2$. Integrating the last inequality, we get the desired energy estimate. Uniqueness follows from the energy estimate. $\square$

2. The first order system. Proof of existence

Introducing the new variables

$$\varphi_{ij} = \partial_t \kappa_{ij}, \qquad \psi_j = -\partial^s \kappa_{sj} + 3\partial_j \kappa,$$

$$\lambda_{lij} = \frac{2}{3}[\partial_l \kappa_{ij} - \partial_{(i}\kappa_{j)l} - \frac{1}{2}\delta_{l(i}\partial^s \kappa_{|s|j)} + \frac{1}{2}\delta_{l(i}\partial_{j)}\kappa + \frac{1}{2}\delta_{ij}\partial^s \kappa_{sl} - \frac{1}{2}\delta_{ij}\partial_l \kappa], \tag{13}$$

we rewrite (9) as (the second and third equation results by differentiating definitions of λ_{lij} and ψ_i):

$$\partial_t \varphi_{ij} = \frac{3}{2}\partial^l \lambda_{lij} + \frac{1}{2}\partial_{(i}\psi_{j)}, \tag{14}$$

$$\partial_t \lambda_{lij} = \frac{2}{3}[\partial_l \varphi_{ij} - \partial_{(i}\varphi_{j)l} - \frac{1}{2}\delta_{l(i}\partial^s \varphi_{|s|j)} + \frac{1}{2}\delta_{l(i}\partial_{j)}\varphi + \frac{1}{2}\delta_{ij}\partial^s \varphi_{sl} - \frac{1}{2}\delta_{ij}\partial_l \varphi], \tag{15}$$

$$\partial_t \psi_j = -\partial^s \varphi_{sj} + 3\partial_j \varphi. \tag{16}$$

Notice, that $\eta_{lij} = \delta_{l(i}\psi_{j)}$; the initial data $\varphi_{ij}(0)$, $\psi_j(0)$, and $\lambda_{li}(0)$ is obtained by substituting $\kappa_{ij}(0)$, $\partial_t \kappa_{ij}(0)$, into the definitions (13).

Equations (13) are added to system (14)–(16) and are enforced as (the artificial) constraint equations. Enforcement of the artificial constraints is necessary since a solution to (14)–(16) solves (6) only if all three equations of (13) hold.

Note that system (14)–(16) is not hyperbolic: its characteristic variables (with complex coefficients) have zero and imaginary speeds only. Moreover, it does not seem possible to have a first order symmetric hyperbolic reduction of (6) unless constraint preservation is invoked. However, we show next that methods of the standard theory of symmetric hyperbolic equations with maximal dissipative conditions can be applied to this system after modification to take the advantage of the fact that the system propagates constraint (2) trivially.

We introduce two constraint quantities

$$\tilde{M}_j = \partial^l \varphi_{lj} - \partial_j \varphi,$$
$$\tilde{N}_j = \frac{3}{2}\partial^l \partial^i \lambda_{lij} + \frac{1}{4}\partial^l \partial_l \psi_j - \frac{1}{4}\partial_j \partial^l \psi_l.$$

The first one has an obvious relationship with constraint (2), $\tilde{M}_j = \partial_t M_j$, while the second one is an easy consequence of (13). Keep in mind that $\tilde{M}_j$ and $\tilde{N}_j$ are not independent variables but rather shorthand notations for combinations of derivatives of φ_{ij}, λ_{lij}, and ψ_j.

The boundary conditions on φ_{ij}, λ_{lij}, and η_j are obtained from (10) as

$$(P_5)_{ij}^{pq}\varphi_{pq} + 3n^p \lambda_{pij} + n_{(i}\psi_{j)} = 0, \qquad \text{on } \partial\Omega. \tag{17}$$

We need to introduce some additional notations. First of all, we define $U = (\varphi_{ij}, \lambda_{lij}, \psi_j)$, and introduce the space $\mathcal{F}$ of all C^∞ vectors U with the property $\partial_t^2 \tilde{M}_j = 0$ and $\partial_t \tilde{N}_j = 0$, that is

$$\mathcal{F} = \{(\varphi_{ij}, \lambda_{lij}, \psi_j) \mid \partial_t^2 \tilde{M}_j = 0, \quad \partial_t \tilde{N}_j := 0\}.$$

The differential constraints in the definition of $\mathcal{F}$ may seem too restrictive, but in fact, if any smooth solution exists to system (10) it must belong to $\mathcal{F}$. Therefore, it is certainly hopeless to look for the solution outside of this class. At the same time, class $\mathcal{F}$ is not empty: let $f, g \in C^\infty$, it is easy to verify that $(\partial_i \partial_j f, 0, \partial_j g) \in \mathcal{F}$.

We introduce operator $\mathcal{L}$ defined by (14)–(16), and rewrite the system in the form

$$\mathcal{L}U = 0. \tag{18}$$

Lemma 6. *The operator $\mathcal{L}$ maps $\mathcal{F}$ into itself. That is if $U \in \mathcal{F}$ then $\mathcal{L}U \in \mathcal{F}$.*

Proof. Let $U = (\varphi_{ij}, \lambda_{lij}, \psi_j) \in \mathcal{F}$, consider $V = \mathcal{L}U$, where explicitly, $V1_{ij} = \partial_t\varphi_{ij} - (\frac{3}{2}\partial^l\lambda_{lij} + \frac{1}{2}\partial_{(i}\psi_{j)})$, $V2_{lij} = \partial_t\lambda_{lij} - (\frac{2}{3}[\partial_l\varphi_{ij} - \cdots])$, $V3_j = \partial_t\psi_j - (-\partial^s\varphi_{sj} + 3\partial_j\varphi)$. It is easier to verify the $\partial\tilde{N}_j = 0$ constraint first. Indeed, by taking the corresponding combination of $V2_{lij}$ and $V3_j$ and simplifying,

$$\frac{3}{2}\partial^l\partial^i V2_{lij} - \frac{1}{4}\partial^l\partial_l V3_j - \frac{1}{4}\partial_j\partial^l V3_l = \partial_t(\frac{3}{2}\partial^l\partial^i\lambda_{lij} - \frac{1}{4}\partial^l\partial_l\psi_j - \frac{1}{4}\partial_j\partial^l\psi_l) = \partial_t\tilde{N}_j = 0.$$

Similarly,

$$\begin{aligned}\partial_t(\partial^l V1_{lj} - \partial_j V1) &= \partial_t^2(\partial^l\varphi_{lj} - \partial_j\varphi) - \partial_t(\frac{3}{2}\partial^l\partial^i\lambda_{lij} - \frac{1}{4}\partial^l\partial_l\psi_j - \frac{1}{4}\partial_j\partial^l\psi_l) \\ &= \partial_t^2 M_j + \partial_t N_j = 0.\end{aligned}$$

□