Quantum Mechanical Healing of Classical Singularities

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Singularities

In a maximal ST,

- A classical singularity exists if there are incomplete geodesics or incomplete curves of bounded acceleration.

- A quantum singularity exists if the evolution of a test wave packet is not uniquely defined by the initial wavepacket, w/o having to add information not present in the wave operator, spacetime metric & manifold alone (ie., one must add boundary conditions at the singularity)

2 categories of singularity

⇒ Question:

Are all classically singular STs quantum mechanically singular as well?

Answer: NO (Horowitz & Marolf (1995))

certain string theory examples
(e.g., certain orbifolds, neg. mass Schw

We will look at this in more detail. In particular, we will consider STs which have a power law form near a timelike singularity at "$r=0$".

Classical singularities

In classical GR, singularities are not part of the ST (the manifold is smooth): they are boundary points in an otherwise maximal ST.

For the timelike & null geodesics (or curves of bounded acceleration), there is an incompleteness, an abrupt ending to the classical particle path.

Singularities in a maximal ST have been divided into 3 types by Ellis & Schmidt (1977):

- QUASIREGULAR (e.g. the 2D cone, the thin idealized cosmic strin
- NON-SCALAR CURVATURE (e.g. whimper cosmologies [Ellis & King (19
- SCALAR CURVATURE (e.g., the center of a Schwarzchild BH, the beginning of a classical Big Bang cosmology)

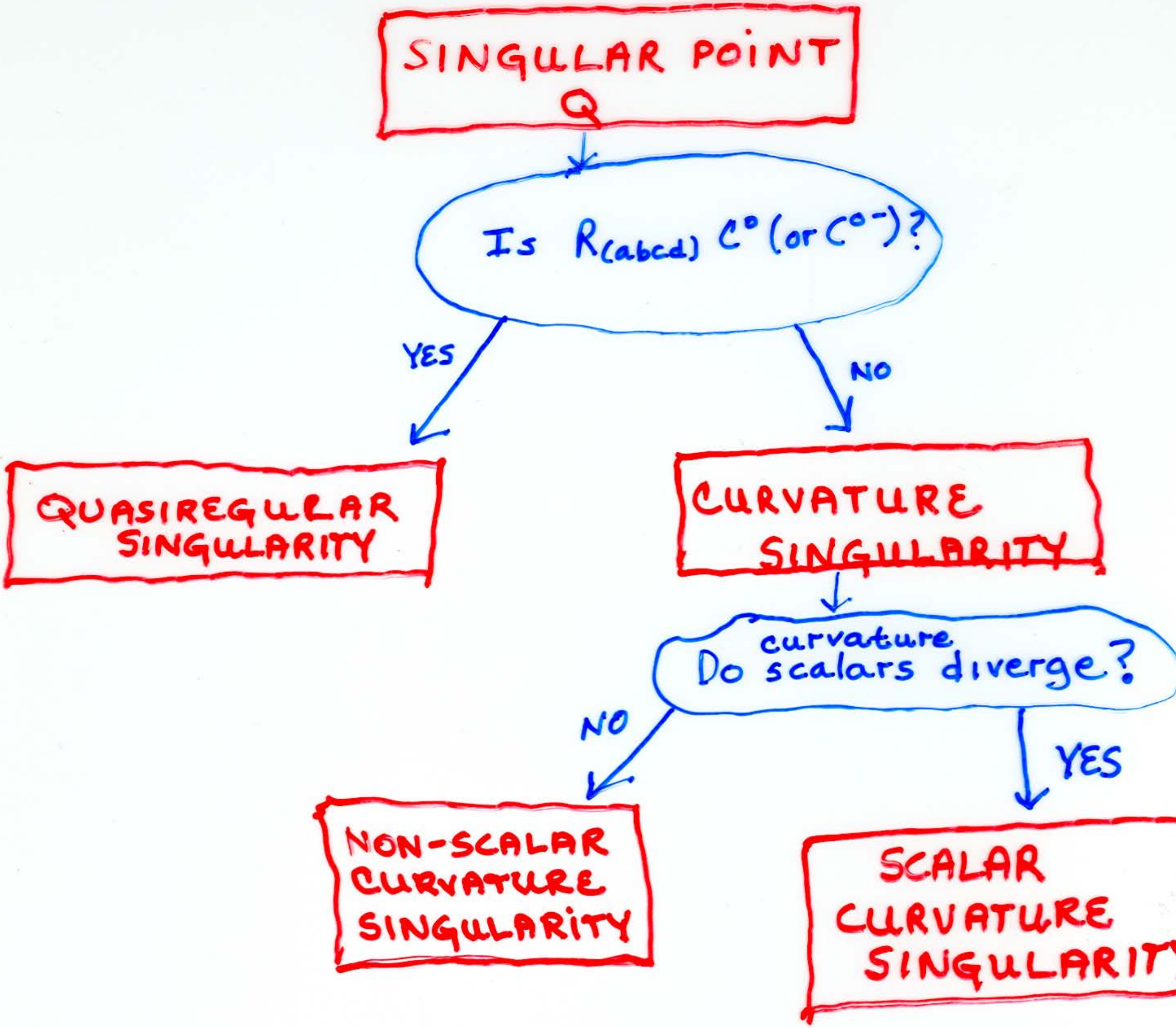
Description of Singularity Types

Mathematical

A singular point q is a **QUASIREGULAR SINGULARITY** if all components of the Riemann tensor R_{abcd} evaluated in an orthonormal frame parallel propagated along an incomplete geodesic ending at q are C^0 (or C^{0-}) (i.e., tend to a finite limit (or are bounded)).

On the other hand, a singular point q is a **CURVATURE SINGULARITY** if some component is not bounded in this way. If all scalars in g_{ab}, the antisymmetric tensor η_{abcd} and R_{abcd} nevertheless tend to a finite limit (or are bounded) the singularity is **NON-SCALAR**, but if any scalar is unbounded, the point q is a **SCALAR CURVATURE SINGULARITY**.

Classical Singularity Types



Description of Singularity Types
→ Physical

The mildest is QUASIREGULAR & the strongest is SCALAR CURVATURE.

At a SCALAR CURVATURE SINGULARITY, physical quantities such as energy density & tidal forces diverge in the frame of all observers who approach the singularity.

At a NON-SCALAR CURVATURE SINGULARITY, there exist curves through each point arbitrarily close to the singularity such that observers on these curves experience perfectly regular tidal forces.

At a QUASIREGULAR SINGULARITY, no observers see or feel physical quantities diverge, even though their worldlines end at the singularity in a finite proper time.

Question:

ARE there **classical singulariti**
("end points" of incomplete geodesics) which are
invisible to real quantum
particles (complete wave packet
specification) ???

STATIC SPACETIMES: with Timelike Singularities

ANSWER

(Horowitz & Marolf 1995)

SINGLE RELATIVISTIC (SCALAR) PARTICLE W/ MASS $M \geq 0$

KLEIN-GORDON EQUATION:

$$\frac{\partial^2 \Psi}{\partial t^2} = - A \Psi$$

$\mathcal{D}(A) = C_0^\infty \subset \mathcal{H} =$

where $A \equiv -VD^i(VD_i) + V^2 m^2$

[timelike Killing field ξ^μ, $V^2 = -\xi^\mu \xi_\mu$,
D_α - spatial covariant derivative Σ (static slice)]

Given $\Psi(0)$ is $\Psi(t)$ UNIQUE?

w/o at SINGU

Yes!

IF A IS ESSENTIALLY SELF-ADJOINT (ESA).

There is a unique, self-adjoint extension A_E which is positive definite so

$$i \frac{d\Psi}{dt} = (A_E)^{1/2} \Psi \Rightarrow$$

UNIQUE SOLN

$$\Psi(t) = \exp(-it(A_E)^{1/2}) \Psi(0)$$

A IS ESSENTIALLY SELF-ADJOINT $\Rightarrow$ Q.M. NONSINGULAR

Quantum Mechanical Mathematical Background

Defn. 1: A **Hilbert space** is an inner product space which is complete.

all Cauchy sequences converge to an element in the space

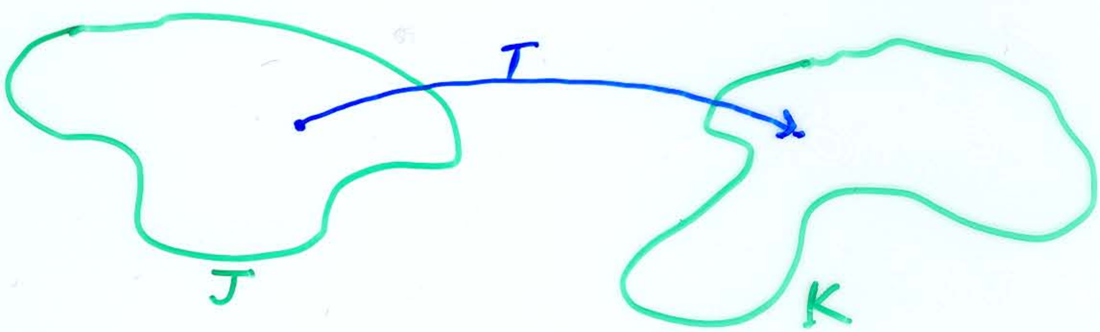
Example: $\ell^2(N)$, the set of all square summable sequences.

$$\ell^2(N) = \{ (x_n)_{n=0}^{\infty} : x_n \in \mathbb{C}, \sum_{n=0}^{\infty} |x_n|^2 < \infty \},$$

w/ inner product

$$\langle (x_n), (y_n) \rangle = \sum_{n=0}^{\infty} x_n \overline{y_n}$$

An operator on a Hilbert space is a linear map from one Hilbert space to another. $T: J \to K$



There is a distinction made between **bounded** and **unbounded** operators. Here we deal w/unbounded operators but first we'll look at bounded ones.

Operators & Adjoints

Defn. 2: Let J, K be Hilbert spaces with $T: J \to K$ a linear map. The following are equivalent:

(1) T is continuous

(2) T is continuous at some point

(3) $\exists\, c > 0$: $\|Tx\| \le c\|x\|$.

In this case, T is called a **bounded operator.**

(Recall: If $\exists$ inner product, $\langle \cdot, \cdot \rangle$, then $\|x\| = \langle x, x \rangle^{1/2}$)

Defn 3.: Let $T: J \to J$ be a bounded operator. The **adjoint** of T is a bounded operator that satisfies $\langle Tx, y \rangle = \langle x, T^* y \rangle \ \forall\, x, y \in J$.

self-adjoint operator: $T = T^*$

We'll need to look at unbounded operators, however.

If T does not satisfy the requirements in Defn. 2, then T is **unbounded.**

Defn. 4: Let J be a Hilbert space with T a linear operator with $\operatorname{dom} T$, $\operatorname{ran} T \subseteq J$. T is **densely defined** if $\overline{\operatorname{dom} T} = J$.

Defn. 5: Consider a densely defined operator $T: J \to J$. It's $\boxed{\text{adjoint, } T^*}$, is defined as follows: $\operatorname{dom} T^* = \{ j \in J : \exists\, f \in J \text{ with } \langle Th, j \rangle = \langle h, f \rangle \ \forall\, h \in \operatorname{dom} T \}$ and $T^* j = f$.

Operators cont.

An unbounded operator T is **self-adjoint** if $T = T^*$ and $\operatorname{dom} T = \operatorname{dom} T^*$

An operator is **essentially self-adjoint** if its closure is self-adjoint. [That is, $T = T^*$ (it's symmetric) and we can make the domains equal.]

$C_0^\infty(R)$ & $\mathcal{L}^2(R)$

The Hilbert space we will be using for our work is $\mathcal{L}^2(R)$.

Start w/ the metric space $C_0^\infty(R)$, the set of all functions from R to C that are differentiable to every order & zero outside a bounded interval. There is an inner product associated w/ this space: for $f, g \in C_0^\infty(R)$,

$$\langle f, g \rangle = \int_{\mathbb{R}} f \bar{g} \, dt.$$

Defn. 6 X is a metric space. Its **completion** is a metric space $\bar{X}$ with the properties:

(1) $\bar{X}$ is complete

(2) There exists an isometry $i : X \to \bar{X}$ so that $i(X)$ is dense in $\bar{X}$.

The completion of $C_0^\infty(R)$ is $\mathcal{L}^2(R)$.

We can think of the elements of $\mathcal{L}^2(R)$ as equivalence classes of functions $f: R \to C$ that are square integrable.

Defn 7: A function f is square integrable on a domain D if $\int_D |f|^2 < \infty$.

We'll usually just say that our Hilbertspace $\mathcal{L}^2(R)$ is the space of square integrable functions.

Q. M. Singular

The operator we will use is the Klein-Gord operator $(\square^2 - M^2)$. Following Horowitz & Marolf we will say that a static ST is Q.M. singular if the spatial portion of this operator is NOT ESSENTIALLY SELF-ADJOINT on C_0^∞ in $\mathcal{L}^2$.

In our work the spatial portion of the K-G operator w/ be separable and we will be left w/ only an "r" equation. We will thus be asking if the assoc. "r" operator is essentially self-adjoint on a $C_0^\infty(0,\infty)$ domain in $\mathcal{L}^2(0,\infty)$.

A useful way to do this is to study the limit point - limit circle behavior of this operator.

WEYL'S LIMIT POINT - LIMIT CIRCLE CRITERION

RE-WRITE RADIAL EQ IN 1D SCHRÖDINGER FORM:

$$H\psi(x) = E\psi(x) \qquad *$$

where generally $x \in (0, \infty)$

& $H = -\frac{d^2}{dx^2} + V(x)$; $\int dx\, \psi^* \psi < \infty$ for L^2

THEOREM (Weyl's):

IF $V(x)$ IS CONTINUOUS REAL-VALUED FUNCTION ON $(0, \infty)$, THEN H IS ESSENTIALLY SELF-ADJOINT ON $C_0^\infty(0, \infty)$ IFF $V(x)$ IS IN THE LIMIT POINT CASE AT BOTH 0 & ∞.

THE POTENTIAL $V(x)$ is in the limit circle case at $x = 0$ if for some, and therefore all, E all solutions of $*$ are square integrable at zero. If $V(x)$ is not in the limit circle case, it is in the limit point case.

(Similar for $\pm\infty$.)

Limit Point - Limit Circle Theorems

Theorem 1. Let $V(x)$ be a continuous real-valued function on $(0,\infty)$ and suppose that there exists a positive differentiable function $M(x)$ so that

(1) $V(x) \geq -M(x)$

(2) $\int_1^\infty [M(x)]^{-1/2}\,dx = \infty$

(3) $M'(x)/(M(x))^{3/2}$ is bounded near ∞.

Then $V(x)$ is in the limit point case at ∞.

Theorem 2. Let $V(x)$ be continuous and positive near zero. If $V(x) \geq (3/4)x^2$ near zero then $V(x)$ is in the limit point case. If for some $\varepsilon > 0$, $V(x) \leq (\frac{3}{4} - \varepsilon)x^2$ near zero, then $V(x)$ is in the limit circle case.

Power Law Metrics

STs as $r \to 0$,

$$ds^2 = -r^\alpha dt^2 + r^\beta dr^2 + \frac{1}{C^2} r^\gamma d\theta^2 + r^\delta dz^2$$

w/ $\alpha, \beta, \gamma, \delta$ constants

Eliminate α by $r \to r^n$.
Get 2 metric types:

Type I: $ds^2 = -r^\beta dt^2 + r^\beta dr^2 + \frac{1}{C^2} r^\gamma d\theta^2 + r^\delta dz^2$ (if $\alpha \neq \beta + 2$)

Type II: $ds^2 = -r^{\beta+2} dt^2 + r^\beta dr^2 + \frac{1}{C^2} r^\gamma d\theta^2 + r^\delta dz^2$ (if $\alpha = \beta + 2$)

Type I metrics are cylindrically symmetric

Type I metrics can be put in form of general cyl. sym. geometry,

$$ds^2 = e^{2(K-U)}(-dt^2 + dr^2) + e^{-2U} W^2 d\phi^2 + e^{2U}(dz + A\, d\phi)^2$$

w/ $A = 0$, $e^{2U} = r^{\delta}$, $e^{2K} = r^{\alpha+\delta}$,

$$W^2 = \frac{1}{c^2} r^{\gamma+\delta}$$

The 2-parameter Levi-Civita STs,

$$ds^2 = -R^{4\sigma} dt^2 + R^{8\sigma^2 - 4\sigma}(dR^2 + dz^2) + 1/c^2\, R^{2-4\sigma} d\theta^2$$

are Type I (if $\sigma \neq 1/2$). The $\sigma = 1/2$ are Type II & flat.

Classical Singularities

Type I :

All curvature scalars vanish if & only if: $\beta = 0$ & (i) $\gamma = \delta = 0$, (ii) $\gamma = 0, \delta = 2$, (iii) $\gamma = 2, \delta = 0$. Each is flat everywhere, although in case (iii) there is a quasiregular (conical) singularity unless $C = 1$, assuming $0 \le \theta \le 2\pi$. Excepting these 3 special cases, Type I STs have a scalar curvature singularity as $r \to 0$ iff $\beta > -2$.

Type II :

All curvature scalars vanish iff $\gamma = \delta = 0$, flat cases. In all other cases, Type II STs have a scalar curvature sing. as $r \to 0$ iff $\beta > -2$.

Quantum Singularity

Goal: to identify the ranges of β, γ, δ, C for which QM's **"heals"** the classical singularities in the sense of Horowitz & Marolf.

For which parameters is there a classical, but no quantum, singularity as $r \to 0$?

Klein-Gordon eqn: $\Box \Phi = M^2 \Phi$

has mode solns $\Phi = e^{-i\omega t} e^{im\theta} e^{ikz} R(r)$

For Type I: $R(r)$ obeys

$$R'' + \left(\frac{\gamma+\delta}{2r}\right) R' + \left[\omega^2 - m^2 C^2 r^{\beta-\gamma} - k^2 r^{\beta-\delta} - M^2 r^{\beta}\right] R = 0$$

For Type II: $R(r)$ obeys

$$R'' + \left(\frac{\gamma+\delta+2}{2r}\right) R' + \left[\frac{\omega^2}{r^2} - m^2 C^2 r^{\beta-\gamma} - k^2 r^{\beta-\delta} - M^2 r^{\beta}\right] R = 0$$

Q.M.s for Type II STs

Change R-eqn to 1D Schrödinger eqn :

$$\frac{d^2\psi}{dx^2} + [E - V(x)]\psi = 0$$

where $r = e^x$ & $R(r) = \sqrt{C}\, e^{-\left(\frac{\gamma+\delta}{4}\right)x}\psi(x)$

Here $E = \omega^2 - \left(\frac{\gamma+\delta}{4}\right)^2$ and

$$V(x) = m^2 C^2 e^{(\beta-\gamma+\delta)x} + k^2 e^{(\beta-\delta+2)x} + M^2 e^{(\beta+2)x}$$

where $-\infty < x < \infty$, At both ends $V(x) > -Kx^2$, so is **limit point** (by Theorem 1) for <u>ALL</u> parameter values.

$\Rightarrow$ **Quantum mechanics "Heals" all classical singularities for Type II STs.**

Q.M. for Type I STs

Change R-eqn to 1D Schrödinger eqn:

$$\frac{d^2\psi}{dx^2} + (E - V(x))\psi = 0$$

where $r = x$, $R = \sqrt{C}\, x^{-\left(\frac{\gamma+\delta}{2}\right)} \psi(x)$ &

normalization is

$$\int dr \sqrt{\frac{-g_3}{g_{00}}}\, R^* R = \int dr \sqrt{\frac{r^{\beta+\delta+\gamma}}{C^2 r^\beta}}\, R^* R$$

$$= \int dx\, \psi^* \psi$$

where $E = \omega^2$ and

$$V(x) = \left(\frac{\gamma+\delta}{4}\right)\left(\frac{\gamma+\delta}{4} - 1\right)\frac{1}{x^2} + m^2 C^2 x^{\beta-\gamma}$$

$$+ k^2 x^{\beta-\delta} + M^2 x^\beta$$

Easy to show $V(x)$ is limit point as $x \to \infty$ (assume also true for exact metric).

For what parameter values is $V(x)$ limit point as $x \to 0$???

Since $V(x) \to c_0 x^n$ as $x \to 0$,

the simple rule is that the potential is

Limit point if $C_0 \geq 0$ & also

$n < -2$, or $n = -2$ & $C_0 \geq \frac{3}{4}$

Limit circle if $C_0 \geq 0$ & also

$n = -2$, and $C_0 < 3/4$, or $n > -2$,

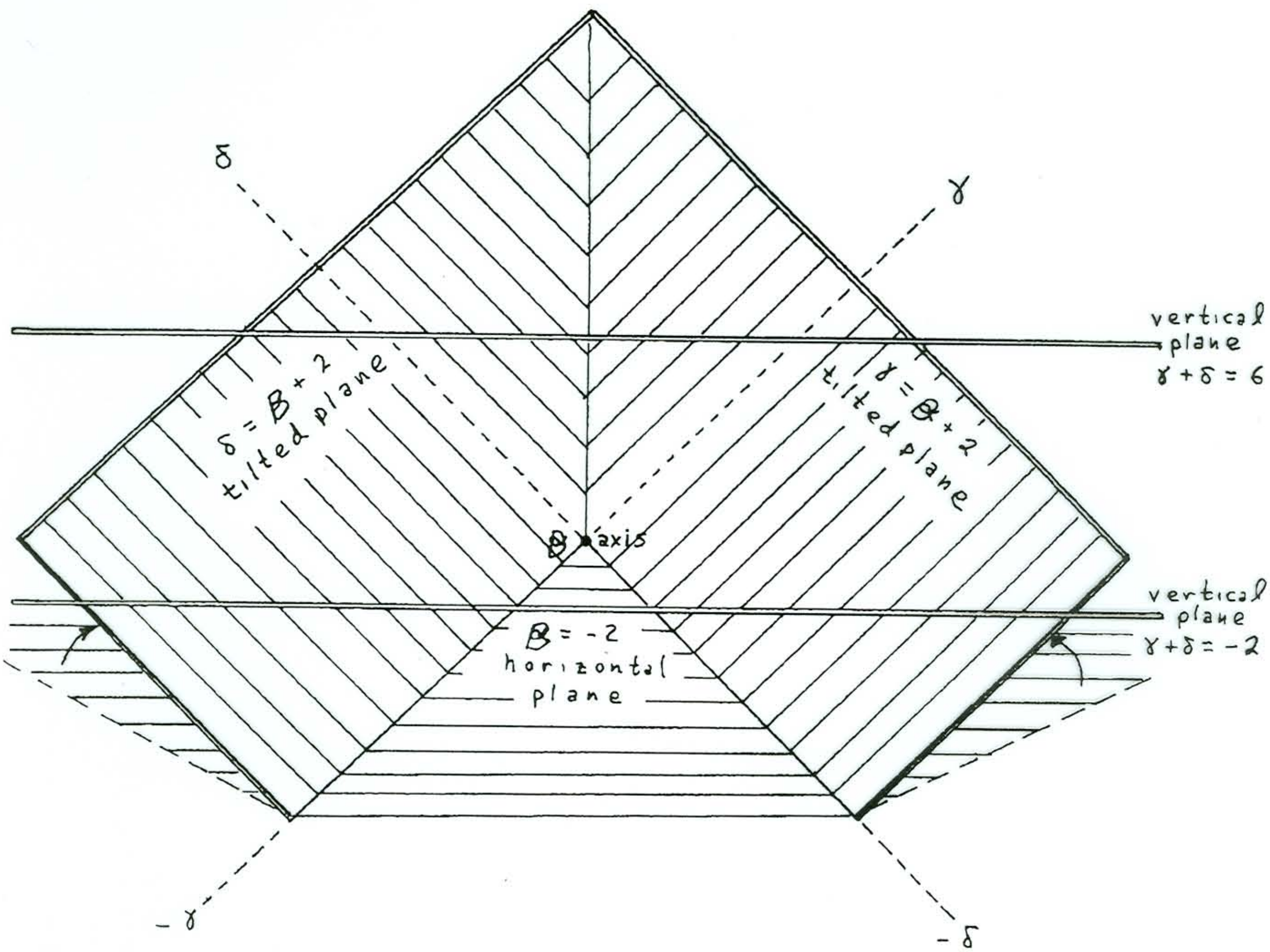
or

if $C_0 < 0$

See 3D plot of

β, γ, δ - space.

Limit Circle Bowl



Except 3 flat STs all Type I STs w/ $\beta > -2$ have classical scalar curvature singularities as $r \to 0$. STs in the limit circle bowl have $V(x)$ limit circle & are Q.M. singular as well. However, classically singular STs w/ $\beta > -2$ lie outside the limit circle bowl & are quantum mechanically "healed". In these STs the classical singularities are in effect invisible to real quantum particles.

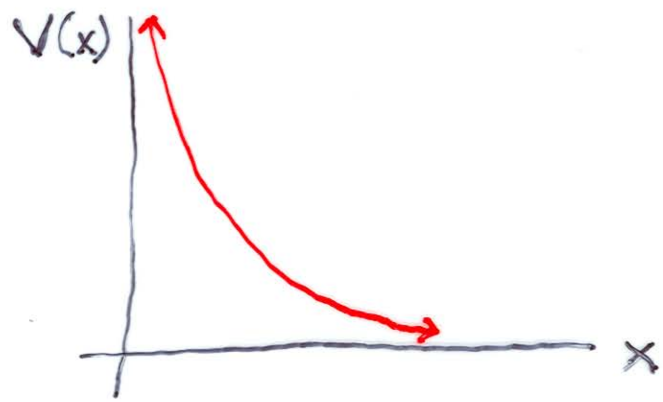
Physical meaning of the LP/LC criterion for Type I metrics

Theorem: If (as $x \to 0$), $V(x) \to c_0 x^n$, the potential is

LP if $c_0 \geq 0$ & also either $n < -2$ or $n = -2$ & $c_0 \geq 3/4$

LC if $c_0 \geq 0$ & also either $n > -2$ or $n = -2$ & $c_0 < 3/4$
or if $c_0 < 0$.

This means that all potentials are LC except those that are sufficiently repulsive near $x = 0$.



The potential can be c_0/x^2 as long as $c_0 \geq 3/4$, or it can be any repulsive potential steeper than $1/x^2$, such as $1/x^3$, $1/x^4$, etc.

If the potential is sufficiently repulsive, one of the solutions diverges so quickly as $x \to 0$ that it is not in $\mathcal{L}^2$ & cannot be used.

With only one function valid near the singularity, the singularity cannot affect the phase of an outgoing scattered wave, so the scattering cross-section is zero: the singularity is invisible to an incoming quantum mechanical particle.

Therefore one can say that even though there may be a classical singularity, if the potential as $x \to 0$ is limit point, there is no quantum singularity present.