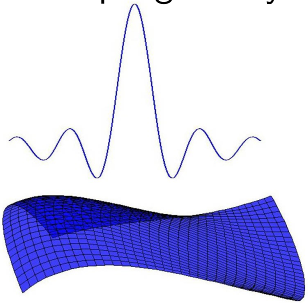
Covariant discretization of space-time using sampling theory



Rob Martin and Achim Kempf University of Waterloo 07/01/07

Outline

Motivation: the problem

Spacetime: discrete or continuous?

- Quantum gravity arguments $\Rightarrow$ existence of a ultra-violet cutoff

Motivation: the problem

Spacetime: discrete or continuous?

- Quantum gravity arguments $\Rightarrow$ existence of a ultra-violet cutoff
- UV divergencies show quantum field theory ill-defined on a continuous manifold

Motivation: the problem

Spacetime: discrete or continuous?

- Quantum gravity arguments $\Rightarrow$ existence of a ultra-violet cutoff
- UV divergencies show quantum field theory ill-defined on a continuous manifold
- QFT becomes well-defined if spacetime a discrete lattice

Motivation: the problem

Spacetime: discrete or continuous?

- Quantum gravity arguments $\Rightarrow$ existence of a ultra-violet cutoff
- UV divergencies show quantum field theory ill-defined on a continuous manifold
- QFT becomes well-defined if spacetime a discrete lattice
- GR lives on a smooth manifold, spacetime discreteness violates manifold symmetries like Lorentz invariance

Motivation: the problem

Spacetime: discrete or continuous?

- Quantum gravity arguments $\Rightarrow$ existence of a ultra-violet cutoff
- UV divergencies show quantum field theory ill-defined on a continuous manifold
- QFT becomes well-defined if spacetime a discrete lattice
- GR lives on a smooth manifold, spacetime discreteness violates manifold symmetries like Lorentz invariance

Q: How to satisfy both QFT and GR ?

Motivation: a proposed solution

- **Idea:** Spacetime continuous but physical fields possess finite ‘density’ of degrees of freedom.

Motivation: a proposed solution

- **Idea:** Spacetime continuous but physical fields possess finite 'density' of degrees of freedom.
- The field values $\{\phi(x_n)\}$ on any 'sufficiently dense' discrete lattice $\Lambda = \{x_n\}_{n\in\mathbb{Z}}$ determine ϕ uniquely
  - *i.e.* can reconstruct $\phi(x)$ everywhere from $\{\phi(x_n)\}_{n\in\mathbb{Z}}$

Motivation: a proposed solution

- **Idea:** Spacetime continuous but physical fields possess finite 'density' of degrees of freedom.
- The field values $\{\phi(x_n)\}$ on any 'sufficiently dense' discrete lattice $\Lambda = \{x_n\}_{n\in\mathbb{Z}}$ determine ϕ uniquely
  - *i.e.* can reconstruct $\phi(x)$ everywhere from $\{\phi(x_n)\}_{n\in\mathbb{Z}}$
- Any physical theory can be defined on the smooth manifold or equivalently on any lattice of 'sufficient density'

Motivation: a proposed solution

- **Idea:** Spacetime continuous but physical fields possess finite 'density' of degrees of freedom.
- The field values $\{\phi(x_n)\}$ on any 'sufficiently dense' discrete lattice $\Lambda = \{x_n\}_{n\in\mathbb{Z}}$ determine ϕ uniquely
  - *i.e.* can reconstruct $\phi(x)$ everywhere from $\{\phi(x_n)\}_{n\in\mathbb{Z}}$
- Any physical theory can be defined on the smooth manifold or equivalently on any lattice of 'sufficient density'
  - No preferred lattice $\Rightarrow$ spacetime symmetries will be preserved

Motivation: a proposed solution

- **Idea:** Spacetime continuous but physical fields possess finite 'density' of degrees of freedom.
- The field values $\{\phi(x_n)\}$ on any 'sufficiently dense' discrete lattice $\Lambda = \{x_n\}_{n\in\mathbb{Z}}$ determine ϕ uniquely
  - *i.e.* can reconstruct $\phi(x)$ everywhere from $\{\phi(x_n)\}_{n\in\mathbb{Z}}$
- Any physical theory can be defined on the smooth manifold or equivalently on any lattice of 'sufficient density'
  - No preferred lattice $\Rightarrow$ spacetime symmetries will be preserved

Consequences for numerical studies

- If physical fields on a manifold have a finite ‘density’ of degrees of freedom
  - Given a DE on a manifold *e.g.* equation of motion for EM field ϕ in black hole spacetime, spacetime is effectively discrete

Consequences for numerical studies

- If physical fields on a manifold have a finite 'density' of degrees of freedom
  - Given a DE on a manifold *e.g.* equation of motion for EM field ϕ in black hole spacetime, spacetime is effectively discrete
  - Derivative operators can be represented as matrix operators on the sample values $\phi(x_n)$ without error

Consequences for numerical studies

- If physical fields on a manifold have a finite 'density' of degrees of freedom
  - Given a DE on a manifold *e.g.* equation of motion for EM field ϕ in black hole spacetime, spacetime is effectively discrete
  - Derivative operators can be represented as matrix operators on the sample values $\phi(x_n)$ without error
- Q: Is there a mathematical theory of function spaces with these special properties?

Consequences for numerical studies

- If physical fields on a manifold have a finite ‘density’ of degrees of freedom
  - Given a DE on a manifold *e.g.* equation of motion for EM field ϕ in black hole spacetime, spacetime is effectively discrete
  - Derivative operators can be represented as matrix operators on the sample values $\phi(x_n)$ without error
- Q: Is there a mathematical theory of function spaces with these special properties?
- A: Yes, it’s called sampling theory

Introduction to sampling theory

Sampling Theory: the study of spaces of functions which are perfectly reconstructible from their values taken on certain discrete sets of points.

- Example: Ω-bandlimited functions
  - Fourier transforms of elements of $L^2[-\Omega, \Omega]$
  - Ω called the bandlimit
- Resources: large body of literature
  - mathematics
  - communication engineering
  - information theory

Bandlimited functions

- $B(\Omega) :=$ Hilbert space of functions bandlimited by Ω

$$f(x) = \int_{-\Omega}^{\Omega} F(w) e^{iwx} dw \quad F \in L^2[-\Omega, \Omega]$$

Bandlimited functions

- $B(\Omega) :=$ Hilbert space of functions bandlimited by Ω

$$f(x) = \int_{-\Omega}^{\Omega} F(w) e^{iwx} dw \quad F \in L^2[-\Omega, \Omega]$$

$$\Rightarrow f(x) = \sum_n f(x_n) \frac{\sin \Omega(x - x_n)}{\Omega(x - x_n)} \quad \text{where} \quad x_n := \frac{n\pi}{\Omega}$$

Bandlimited functions

- $B(\Omega)$:= Hilbert space of functions bandlimited by Ω

$$f(x) = \int_{-\Omega}^{\Omega} F(w) e^{iwx} dw \quad F \in L^2[-\Omega, \Omega]$$

- Details: ON basis of $L^2[-\Omega, \Omega]$: $\{b_{x_n}\}_{n\in\mathbb{Z}}$, $b_{x_n}(w) = \frac{1}{\sqrt{2\Omega}} e^{ix_n w}$, where $x_n := \frac{n\pi}{\Omega}$

$$F = \sum_n (F, b_{x_n}) b_{x_n} \quad \text{where} \quad (F, b_{x_n}) = \frac{1}{\sqrt{2\Omega}} \int_{-\Omega}^{\Omega} F(w) e^{iwx_n} dw = \frac{1}{\sqrt{2\Omega}} f(x_n) \tag{1}$$

$$f = \mathcal{F}^{-1} F = \sum_n (F, b_{x_n}) \mathcal{F}^{-1} b_{x_n} \quad \text{where} \quad \mathcal{F}^{-1} b_{x_n} = 2\Omega \frac{\sin \Omega(x - x_n)}{\Omega(x - x_n)} \tag{2}$$

$$\Rightarrow f(x) = \sum_n f(x_n) \frac{\sin \Omega(x - x_n)}{\Omega(x - x_n)} \quad \text{where} \quad x_n := \frac{n\pi}{\Omega}$$

Discretized derivatives

- For $B(\Omega)$, $\mathbb{R}$ is 'effectively discrete' since any $f \in B(\Omega)$ is perfectly reconstructible from $\{f(x_n)\}$, $x_n = \frac{n\pi}{\Omega}$.

Discretized derivatives

- For $B(\Omega)$, $\mathbb{R}$ is 'effectively discrete' since any $f \in B(\Omega)$ is perfectly reconstructible from $\{f(x_n)\}$, $x_n = \frac{n\pi}{\Omega}$.
- Derivative operator on $B(\Omega)$ is automatically discretized, can be represented (without error) as matrix acting on $\{f(x_n)\}_{n\in\mathbb{Z}}$.

Discretized derivatives

- For $B(\Omega)$, $\mathbb{R}$ is 'effectively discrete' since any $f \in B(\Omega)$ is perfectly reconstructible from $\{f(x_n)\}$, $x_n = \frac{n\pi}{\Omega}$.
- Derivative operator on $B(\Omega)$ is automatically discretized, can be represented (without error) as matrix acting on $\{f(x_n)\}_{n\in\mathbb{Z}}$.
- $f \in B(\Omega) \Rightarrow f' \in B(\Omega)$

$$f'(x) = \sum_n f'(x_n) \frac{\sin \Omega(x - x_n)}{\Omega(x - x_n)} \text{ where } f'(x_n) = \sum_m D(x_m, x_n) f(x_m)$$

Discretized derivatives

- For $B(\Omega)$, $\mathbb{R}$ is 'effectively discrete' since any $f \in B(\Omega)$ is perfectly reconstructible from $\{f(x_n)\}$, $x_n = \frac{n\pi}{\Omega}$.
- Derivative operator on $B(\Omega)$ is automatically discretized, can be represented (without error) as matrix acting on $\{f(x_n)\}_{n\in\mathbb{Z}}$.
- $f \in B(\Omega) \Rightarrow f' \in B(\Omega)$

$$f'(x) = \sum_n f'(x_n) \frac{\sin \Omega(x - x_n)}{\Omega(x - x_n)} \text{ where } f'(x_n) = \sum_m D(x_m, x_n) f(x_m)$$

$$f(x) = \sum_n f(x_n) \frac{\sin \Omega(x - x_n)}{\Omega(x - x_n)} \rightarrow f'(x) = \sum_n f(x_n) \frac{d}{dx} \left(\frac{\sin \Omega(x - x_n)}{\Omega(x - x_n)} \right)$$

$$D(x, x_n) := \frac{d}{dx} \left(\frac{\sin \Omega(x - x_n)}{\Omega(x - x_n)} \right) = \frac{(x - x_n)\cos \Omega(x - x_n) - \sin \Omega(x - x_n)}{\Omega(x - x_n)^2}$$

Sampling Lattices for $B(\Omega)$

- $\Lambda := \{y_n\}_{n\in\mathbb{Z}} \subset \mathbb{R}$ is called uniformly discrete if $\exists\epsilon > 0$ s.t. $|y_n - y_m| > \epsilon$

Sampling Lattices for $B(\Omega)$

- $\Lambda := \{y_n\}_{n\in\mathbb{Z}} \subset \mathbb{R}$ is called uniformly discrete if $\exists \epsilon > 0$ s.t. $|y_n - y_m| > \epsilon$

Definition

Λ is called a set of sampling if the linear map $T : B(\Omega) \to l^2(\mathbb{Z})$ where $Tf = \{f(y_n)\}_{n\in\mathbb{Z}}$ is bounded above and below.

Sampling Lattices for $B(\Omega)$

- $\Lambda := \{y_n\}_{n\in\mathbb{Z}} \subset \mathbb{R}$ is called uniformly discrete if $\exists\epsilon > 0$ s.t. $|y_n - y_m| > \epsilon$

Definition

Λ is called a set of sampling if the linear map $T : B(\Omega) \to l^2(\mathbb{Z})$ where $Tf = \{f(y_n)\}_{n\in\mathbb{Z}}$ is bounded above and below.

- *i.e.* if $\Lambda = \{y_n\}$ is a set of sampling then any $f \in B(\Omega)$ is reconstructible from its sample values $\{f(y_n)\}$

Any sufficiently dense lattice is sampling

- Any lattice of sufficient density is sampling

Any sufficiently dense lattice is sampling

- Any lattice of sufficient density is sampling
  $\Lambda = \{y_n\}$
  - $n_-(r) :=$ min. number of y_n in any interval of length r
  - $D_-(\Lambda) := \lim_{r\to\infty} \frac{n_-(r)}{r}$

Any sufficiently dense lattice is sampling

- Any lattice of sufficient density is sampling
  $\Lambda = \{y_n\}$
  - $n_-(r) :=$ min. number of y_n in any interval of length r
  - $D_-(\Lambda) := \lim_{r\to\infty} \frac{n_-(r)}{r}$

Theorem

(Beurling/Duffin&Schaeffer)
(a) If Λ a set of sampling then $D_-(\Lambda) \geq \frac{\Omega}{\pi}$
(b) Λ is a set of sampling if $D_-(\Lambda) > \frac{\Omega}{\pi}$

Sampling lattices for $B(S)$

- More generally, $B(S) :=$ subspace of $L^2(\mathbb{R}^n)$ whose elements are Fourier transforms of vectors in $L^2(S)$. $S \subset \mathbb{R}^n$ compact.

$$f(x) = \int_S F(w) e^{iwx} dw \quad F \in L^2(S)$$

Sampling lattices for $B(S)$

- More generally, $B(S) :=$ subspace of $L^2(\mathbb{R}^n)$ whose elements are Fourier transforms of vectors in $L^2(S)$. $S \subset \mathbb{R}^n$ compact.

$$f(x) = \int_S F(w) e^{iwx} dw \quad F \in L^2(S)$$

Theorem

(Landau) If Λ is a set of sampling for $B(S)$ then $D_-(\Lambda) \geq \frac{\mu(S)}{(2\pi)^n}$

Minimum possible sample density $\propto$ bandwidth volume

A generally covariant bandlimit

- A bandlimit is a cutoff on the spectrum of the Laplacian in $L^2(M)$ on the manifold $M = \mathbb{R}$

A generally covariant bandlimit

- A bandlimit is a cutoff on the spectrum of the Laplacian in $L^2(M)$ on the manifold $M = \mathbb{R}$
- $-\Delta = -\frac{d^2}{dx^2}$, $-\frac{d^2}{dx^2} e^{\pm iwx} = w^2 e^{\pm iwx}$

A generally covariant bandlimit

- A bandlimit is a cutoff on the spectrum of the Laplacian in $L^2(M)$ on the manifold $M = \mathbb{R}$
- $-\Delta = -\frac{d^2}{dx^2}$, $-\frac{d^2}{dx^2} e^{\pm iwx} = w^2 e^{\pm iwx}$
- $B(\Omega) :=$ subspace spanned by e^{iwx} for $|w| \leq \Omega^2$

$$f(x) = \int_{-\Omega}^{\Omega} F(w) e^{iwx} dw \;\Leftrightarrow\; f \in B(\Omega)$$

A generally covariant bandlimit

- A bandlimit is a cutoff on the spectrum of the Laplacian in $L^2(M)$ on the manifold $M = \mathbb{R}$
- $-\Delta = -\frac{d^2}{dx^2}, \ -\frac{d^2}{dx^2} e^{\pm iwx} = w^2 e^{\pm iwx}$
- $B(\Omega) :=$ subspace spanned by e^{iwx} for $|w| \leq \Omega^2$

$$f(x) = \int_{-\Omega}^{\Omega} F(w) e^{iwx} dw \ \Leftrightarrow \ f \in B(\Omega)$$

Definition

(Pesenson) $B(M, \Omega) :=$ *subspace of* $L^2(M)$ *spanned by eigenfunctions to* $-\Delta$ *(or* $-\Box$*) with eigenvalues* Λ *obeying* $|\Lambda| \leq \Omega^2$.

A generally covariant UV cutoff

- Idea: impose this covariant bandlimit/UV cutoff on physical fields on spacetime

A generally covariant UV cutoff

- Idea: impose this covariant bandlimit/UV cutoff on physical fields on spacetime
- *i.e.* Given a spacetime M, we restrict the set of physical field to be the subspace $B(M, \Omega)$.

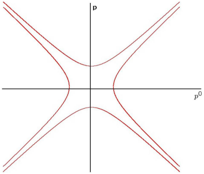
A generally covariant UV cutoff

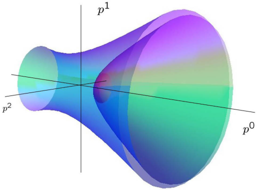
- Idea: impose this covariant bandlimit/UV cutoff on physical fields on spacetime
- *i.e.* Given a spacetime M, we restrict the set of physical field to be the subspace $B(M, \Omega)$.
- Investigate: as in the case $M = \mathbb{R}$, do elements of $B(M, \Omega)$ have a finite 'density' of degrees of freedom

Sampling in Minkowski spacetime

- D'Alembertian: $-\Box = -\frac{d^2}{dt^2} + \Delta$, eigenfunctions: $e^{i(p^0 t - \mathbf{p}\cdot\mathbf{x})}$

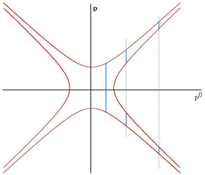
Sampling in Minkowski spacetime

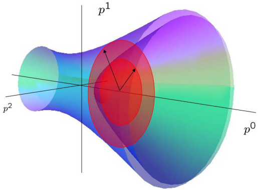




- D'Alembertian: $-\Box = -\frac{d^2}{dt^2} + \Delta$, eigenfunctions: $e^{i(p^0 t - \mathbf{p}\cdot\mathbf{x})}$
- spectral cutoff/covariant bandlimit: $|(p^0)^2 - \mathbf{p}^2| \leq \Omega^2$

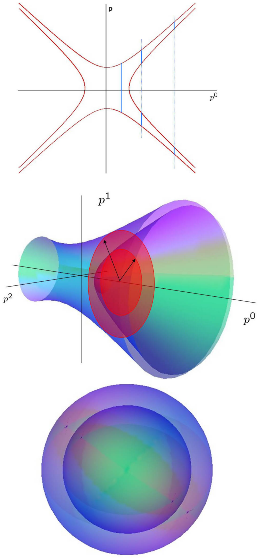
Sampling in Minkowski spacetime





- D'Alembertian: $-\Box = -\frac{d^2}{dt^2} + \Delta$, eigenfunctions: $e^{i(p^0 t - \mathbf{p}\cdot\mathbf{x})}$
- spectral cutoff/covariant bandlimit: $|(p^0)^2 - \mathbf{p}^2| \leq \Omega^2$
- Given $\phi \in B(M, \Omega)$, each fixed temporal mode $\Phi(p^0, x)$ (p^0 fixed) is spatially bandlimited.

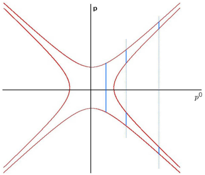
Sampling in Minkowski spacetime

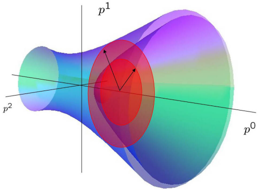


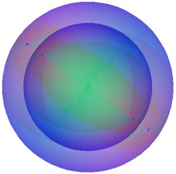
- D'Alembertian: $-\Box = -\frac{d^2}{dt^2} + \Delta$, eigenfunctions: $e^{i(p^0 t - \mathbf{p}\cdot\mathbf{x})}$
- spectral cutoff/covariant bandlimit: $|(p^0)^2 - \mathbf{p}^2| \leq \Omega^2$
- Given $\phi \in B(M, \Omega)$, each fixed temporal mode $\Phi(p^0, x)$ (p^0 fixed) is spatially bandlimited.
- spatial bandwidth volume of $\Phi(p^0, x)$:

$$\sqrt{\Omega^2 + (p^0)^2} \geq |\mathbf{p}| \geq \sqrt{(p^0)^2 - \Omega^2}$$

Sampling in Minkowski spacetime





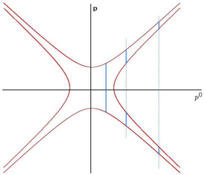


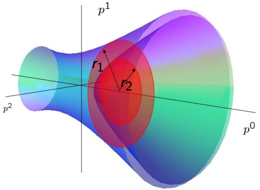
- D'Alembertian: $-\Box = -\frac{d^2}{dt^2} + \Delta$, eigenfunctions: $e^{i(p^0 t - \mathbf{p}\cdot\mathbf{x})}$
- spectral cutoff/covariant bandlimit: $|(p^0)^2 - \mathbf{p}^2| \leq \Omega^2$
- Given $\phi \in B(M, \Omega)$, each fixed temporal mode $\Phi(p^0, x)$ (p^0 fixed) is spatially bandlimited.
- spatial bandwidth volume of $\Phi(p^0, x)$:

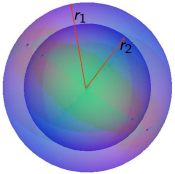
$$\sqrt{\Omega^2 + (p^0)^2} \geq |\mathbf{p}| \geq \sqrt{(p^0)^2 - \Omega^2}$$

  - vanishes as $p^0 \to \infty$ in $1+1$D, high temporal frequencies have few spatial degrees of freedom

Sampling in Minkowski spacetime





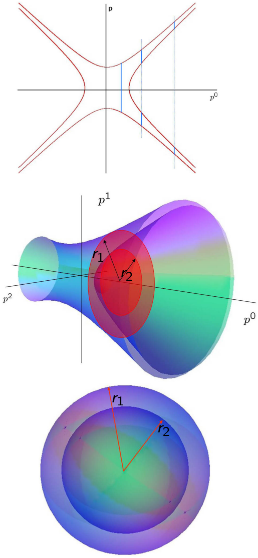


- D'Alembertian: $-\Box = -\frac{d^2}{dt^2} + \Delta$, eigenfunctions: $e^{i(p^0 t - \mathbf{p}\cdot\mathbf{x})}$
- spectral cutoff/covariant bandlimit: $|(p^0)^2 - \mathbf{p}^2| \leq \Omega^2$
- Given $\phi \in B(M,\Omega)$, each fixed temporal mode $\Phi(p^0, x)$ (p^0 fixed) is spatially bandlimited.
- spatial bandwidth volume of $\Phi(p^0, x)$:

$$r_1 := \sqrt{\Omega^2 + (p^0)^2} \geq |\mathbf{p}| \geq \sqrt{(p^0)^2 - \Omega^2} := r_2$$

  - vanishes as $p^0 \to \infty$ in $1+1$D, high temporal frequencies have few spatial degrees of freedom
  - stays constant in $1+2D$

Sampling in Minkowski spacetime

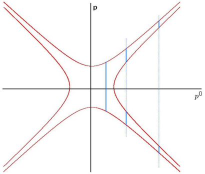


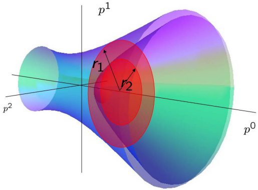
- D'Alembertian: $-\Box = -\frac{d^2}{dt^2} + \Delta$, eigenfunctions: $e^{i(p^0 t - \mathbf{p}\cdot\mathbf{x})}$
- spectral cutoff/covariant bandlimit: $|(p^0)^2 - \mathbf{p}^2| \leq \Omega^2$
- Given $\phi \in B(M, \Omega)$, each fixed temporal mode $\Phi(p^0, x)$ (p^0 fixed) is spatially bandlimited.
- spatial bandwidth volume of $\Phi(p^0, x)$:

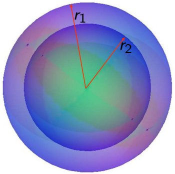
$$r_1 := \sqrt{\Omega^2 + (p^0)^2} \geq |\mathbf{p}| \geq \sqrt{(p^0)^2 - \Omega^2} := r_2$$

  - vanishes as $p^0 \to \infty$ in $1+1$D, high temporal frequencies have few spatial degrees of freedom
  - stays constant in $1 + 2D$
  - diverges in $1 + 3D$.

Sampling in Minkowski spacetime





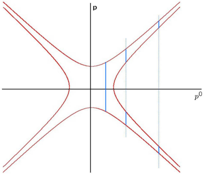


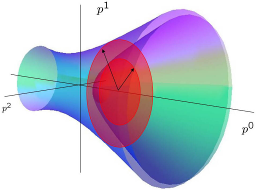
- D'Alembertian: $-\Box = -\frac{d^2}{dt^2} + \Delta$, eigenfunctions: $e^{i(p^0 t - \mathbf{p}\cdot\mathbf{x})}$
- spectral cutoff/covariant bandlimit: $|(p^0)^2 - \mathbf{p}^2| \leq \Omega^2$
- Given $\phi \in B(M, \Omega)$, each fixed temporal mode $\Phi(p^0, x)$ (p^0 fixed) is spatially bandlimited.
- spatial bandwidth volume of $\Phi(p^0, x)$:

$$r_1 := \sqrt{\Omega^2 + (p^0)^2} \geq |\mathbf{p}| \geq \sqrt{(p^0)^2 - \Omega^2} := r_2$$

  - vanishes as $p^0 \to \infty$ in $1+1$D, high temporal frequencies have few spatial degrees of freedom
  - stays constant in $1 + 2D$
  - diverges in $1 + 3D$.

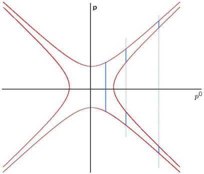
Sampling in Minkowski spacetime II

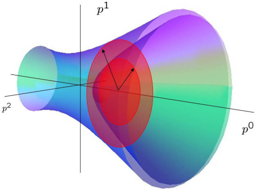




- For $1+1D$ or $1+2D$, spatial bandwidth volume for any fixed p^0 is bounded above

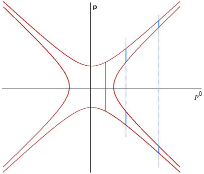
Sampling in Minkowski spacetime II

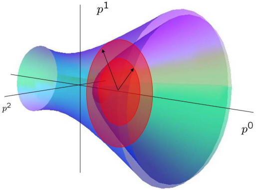




- For $1+1D$ or $1+2D$, spatial bandwidth volume for any fixed p^0 is bounded above
  - one can find sufficiently dense spatial lattices $\Lambda := \{\mathbf{x}_n\}_{n\in\mathbb{Z}}$ which are sampling for every fixed temporal mode p^0.

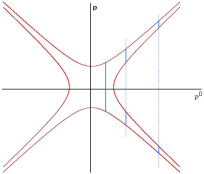
Sampling in Minkowski spacetime II

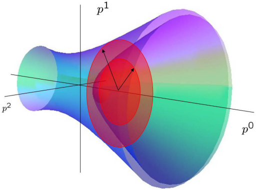




- For $1+1D$ or $1+2D$, spatial bandwidth volume for any fixed p^0 is bounded above
  - one can find sufficiently dense spatial lattices $\Lambda := \{\mathbf{x}_n\}_{n\in\mathbb{Z}}$ which are sampling for every fixed temporal mode p^0.
  - every temporal mode $\Phi(p^0, \mathbf{x})$ can be reconstructed from $\{\Phi(p^0, \mathbf{x_n})\}_{n\in\mathbb{Z}}$

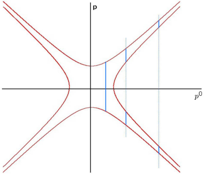
Sampling in Minkowski spacetime II

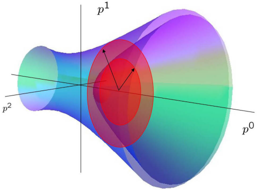




- For $1+1D$ or $1+2D$, spatial bandwidth volume for any fixed p^0 is bounded above
  - one can find sufficiently dense spatial lattices $\Lambda := \{\mathbf{x}_n\}_{n\in\mathbb{Z}}$ which are sampling for every fixed temporal mode p^0.
  - every temporal mode $\Phi(p^0, \mathbf{x})$ can be reconstructed from $\{\Phi(p^0, \mathbf{x_n})\}_{n\in\mathbb{Z}}$
  - $\Rightarrow$ every $\phi \in B(M, \Omega)$ is reconstructible from $\{\phi(t, \mathbf{x_n})\}_{n\in\mathbb{Z}}$

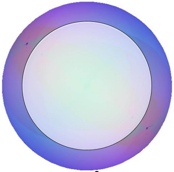
Sampling in Minkowski spacetime II





- For $1+1D$ or $1+2D$, spatial bandwidth volume for any fixed p^0 is bounded above
  - one can find sufficiently dense spatial lattices $\Lambda := \{\mathbf{x}_n\}_{n\in\mathbb{Z}}$ which are sampling for every fixed temporal mode p^0.
  - every temporal mode $\Phi(p^0, \mathbf{x})$ can be reconstructed from $\{\Phi(p^0, \mathbf{x_n})\}_{n\in\mathbb{Z}}$
  - $\Rightarrow$ every $\phi \in B(M, \Omega)$ is reconstructible from $\{\phi(t, \mathbf{x_n})\}_{n\in\mathbb{Z}}$

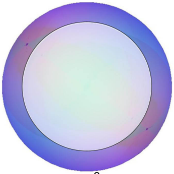
Sampling in Minkowski spacetime III



$\mathrm{BW\ Vol}(p^0) \overset{p^0\to\infty}{\longrightarrow} \infty$

- For $1+3D$ there is no upper bound on spatial bandwidth volume for fixed p^0

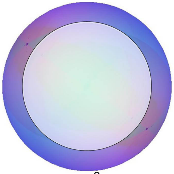
Sampling in Minkowski spacetime III



$\text{BW Vol}(p^0) \overset{p^0 \to \infty}{\longrightarrow} \infty$

- For $1 + 3D$ there is no upper bound on spatial bandwidth volume for fixed p^0
  - Landau's theorem $\Rightarrow$ there is no spatial lattice which is sampling for every p^0.

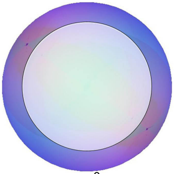
Sampling in Minkowski spacetime III



BW $\mathrm{Vol}(p^0) \overset{p^0 \to \infty}{\longrightarrow} \infty$

- For $1+3D$ there is no upper bound on spatial bandwidth volume for fixed p^0
    - Landau's theorem $\Rightarrow$ there is no spatial lattice which is sampling for every p^0.
- can show that $\exists$ sets $\Lambda := \{x_n^1, x_m^2\}_{n,m\in\mathbb{Z}}$ s.t. one can reconstruct $\phi \in B(M,\Omega)$ everywhere from $\{\phi(t, x_n^1, x_m^2, x^3)\}_{n,m\in\mathbb{Z}\ t,x\in\mathbb{R}}$

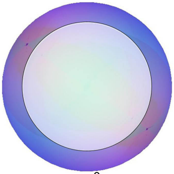
Sampling in Minkowski spacetime III



BW $\mathrm{Vol}(p^0) \xrightarrow{p^0 \to \infty} \infty$

- For $1 + 3D$ there is no upper bound on spatial bandwidth volume for fixed p^0
    - Landau's theorem $\Rightarrow$ there is no spatial lattice which is sampling for every p^0.
- can show that $\exists$ sets $\Lambda := \{x_n^1, x_m^2\}_{n,m\in\mathbb{Z}}$ s.t. one can reconstruct $\phi \in B(M, \Omega)$ everywhere from $\{\phi(t, x_n^1, x_m^2, x^3)\}_{n,m\in\mathbb{Z}\ t,x\in\mathbb{R}}$
    - covariantly bandlimited fields have finite density of degrees of freedom on $2D$ hypersurfaces - useful for holography

Sampling in Minkowski spacetime III



BW $\mathrm{Vol}(p^0) \stackrel{p^0\to\infty}{\longrightarrow} \infty$

- For $1+3D$ there is no upper bound on spatial bandwidth volume for fixed p^0
  - Landau's theorem $\Rightarrow$ there is no spatial lattice which is sampling for every p^0.
- can show that $\exists$ sets $\Lambda := \{x_n^1, x_m^2\}_{n,m\in\mathbb{Z}}$ s.t. one can reconstruct $\phi \in B(M,\Omega)$ everywhere from $\{\phi(t, x_n^1, x_m^2, x^3)\}_{n,m\in\mathbb{Z}\ t,x\in\mathbb{R}}$
  - covariantly bandlimited fields have finite density of degrees of freedom on $2D$ hypersurfaces - useful for holography
- seek similar results on more general spacetimes

Sampling theory in FRW spacetimes

- For simplicity consider $1+1D$. Line element: $ds^2 = -dt^2 + a(t)^2 dx^2 = a^2(\eta)(-d\eta^2 + dx^2)$ where $\eta :=$ conformal time

Sampling theory in FRW spacetimes

- For simplicity consider $1+1D$. Line element: $ds^2 = -dt^2 + a(t)^2 dx^2 = a^2(\eta)(-d\eta^2 + dx^2)$ where $\eta :=$ conformal time
- d'Alembertian: $-\Box := a^{-2}(\eta)\left(-\partial_\eta^2 + \partial_x^2\right) = -a^{-2}(\eta)\left(\partial_\eta^2 + k^2\right)$

Sampling theory in FRW spacetimes

- For simplicity consider $1+1D$. Line element: $ds^2 = -dt^2 + a(t)^2 dx^2 = a^2(\eta)(-d\eta^2 + dx^2)$ where $\eta :=$ conformal time
- d'Alembertian: $-\Box := a^{-2}(\eta)\left(-\partial_\eta^2 + \partial_x^2\right) = -a^{-2}(\eta)\left(\partial_\eta^2 + k^2\right)$
- Seek (as in Minkowski case) a temporal sampling theorem for each fixed spatial mode and visa versa

Sampling theory in FRW spacetimes

- For simplicity consider $1+1D$. Line element: $ds^2 = -dt^2 + a(t)^2 dx^2 = a^2(\eta)(-d\eta^2 + dx^2)$ where $\eta :=$ conformal time
- d'Alembertian: $-\Box := a^{-2}(\eta)\left(-\partial_\eta^2 + \partial_x^2\right) = -a^{-2}(\eta)\left(\partial_\eta^2 + k^2\right)$
- Seek (as in Minkowski case) a temporal sampling theorem for each fixed spatial mode and visa versa

de Sitter with finite end time

- de Sitter with finite end time: $a(t) = e^t$ where $t \in (-\infty, 0]$ *i.e.* $\eta = e^{-t}$ and $\eta \in [1, \infty)$.

de Sitter with finite end time

- de Sitter with finite end time: $a(t) = e^t$ where $t \in (-\infty, 0]$ *i.e.* $\eta = e^{-t}$ and $\eta \in [1, \infty)$.
- d'Alembertian: $-\Box = -\eta^2 \left(\partial_\eta^2 + k^2\right)$ on $L^2\left([1,\infty) \times (-\infty,\infty); \eta^{-2} d\eta dk\right)$.

de Sitter with finite end time

- de Sitter with finite end time: $a(t) = e^t$ where $t \in (-\infty, 0]$ *i.e.* $\eta = e^{-t}$ and $\eta \in [1, \infty)$.
- d'Alembertian: $-\Box = -\eta^2 \left(\partial_\eta^2 + k^2\right)$ on $L^2 \left([1, \infty) \times (-\infty, \infty); \eta^{-2} d\eta dk\right)$.
- bandlimit/spectral cutoff on $-\Box$: $|\lambda| \leq \Omega^2$, $\lambda \in \sigma(-\Box)$. *i.e.* consider $P_{[0,\Omega^2]}(-\Box)\mathcal{H}$.

Sampling in de Sitter: Preliminary results

- Let $\phi \in B(M, \Omega)$, Φ its spatial F.T.
- each fixed spatial mode, $\Phi(\eta, k)$ for each fixed $k > 0$ belongs to a finite dimensional subspace of dimension N_k

Sampling in de Sitter: Preliminary results

- Let $\phi \in B(M, \Omega)$, Φ its spatial F.T.
- each fixed spatial mode, $\Phi(\eta, k)$ for each fixed $k > 0$ belongs to a finite dimensional subspace of dimension N_k
- the larger the spatial mode k - the fewer the number of temporal degrees of freedom.

Sampling in de Sitter: Preliminary results

- Let $\phi \in B(M, \Omega)$, Φ its spatial F.T.
- each fixed spatial mode, $\Phi(\eta, k)$ for each fixed $k > 0$ belongs to a finite dimensional subspace of dimension N_k
- the larger the spatial mode k - the fewer the number of temporal degrees of freedom.
    - $N_k \to 2$ as $k \to \infty$, $N_k \to \infty$ as $k \to 0$.

Sampling in de Sitter: Preliminary results

- Let $\phi \in B(M, \Omega)$, Φ its spatial F.T.
- each fixed spatial mode, $\Phi(\eta, k)$ for each fixed $k > 0$ belongs to a finite dimensional subspace of dimension N_k
- the larger the spatial mode k - the fewer the number of temporal degrees of freedom.
    - $N_k \to 2$ as $k \to \infty$, $N_k \to \infty$ as $k \to 0$.
- Can show $\Phi(\eta, k)$ reconstructible from values $\{\Phi(\eta_j, k)\}_{j=1}^{N_k}$ for certain choices of $\{\eta_j\}_{j=1}^{N_k}$.

de Sitter: Preliminary results

- Can show for $k = 0$, $\Phi(\eta, 0)$ obeys the following sampling formula

$$\Phi(\eta, 0) = \sum_{n=0}^{\infty} \frac{\Omega}{\sqrt{\eta_n}} \Phi(\eta_n, 0) K(\eta_n, \eta) \quad \text{where} \quad \eta_n := e^{\frac{n\pi}{\Omega}} \qquad (3)$$

de Sitter: Preliminary results

- Can show for $k = 0$, $\Phi(\eta, 0)$ obeys the following sampling formula

$$\Phi(\eta, 0) = \sum_{n=0}^{\infty} \frac{\Omega}{\sqrt{\eta_n}} \Phi(\eta_n, 0) K(\eta_n, \eta) \quad \text{where} \quad \eta_n := e^{\frac{n\pi}{\Omega}} \qquad (3)$$

- Details:

$$K(\eta, y) = \frac{\sin \frac{\Omega}{2}(\ln \eta + \ln y)}{\frac{\Omega}{2}(\ln \eta + \ln y)} \cos \frac{\Omega}{2}(\ln \eta + \ln y) + \frac{\sin \frac{\Omega}{2}(\ln(\eta) - \ln y)}{\frac{\Omega}{2}(\ln(\eta) - \ln y)} cos \frac{\Omega}{2}(\ln \eta - \ln y) \qquad (4)$$

$$P_{[-\Omega, \Omega]}(-\Box_0) f(\eta) = \int_1^\infty f(\eta') K(\eta, \eta') \frac{\Omega \sqrt{\eta}}{\pi (\eta')^{3/2}} d\eta' \qquad (5)$$

Sampling in de Sitter: Further Questions

- Conjecture that there exists an overall temporal sampling formula that reconstructs $\phi \in B(M, \Omega)$ from $\{\phi(\eta_j, x)\}_{j\in\mathbb{N}, x\in\mathbb{R}}$.

Sampling in de Sitter: Further Questions

- Conjecture that there exists an overall temporal sampling formula that reconstructs $\phi \in B(M, \Omega)$ from $\{\phi(\eta_j, x)\}_{j\in\mathbb{N}, x\in\mathbb{R}}$.
- Full de Sitter: have worked out expressions for the projectors $P_{[-\Omega,\Omega]}(-\Box_k)$ but still deriving sampling formulas

Sampling in de Sitter: Further Questions

- Conjecture that there exists an overall temporal sampling formula that reconstructs $\phi \in B(M, \Omega)$ from $\{\phi(\eta_j, x)\}_{j\in\mathbb{N}, x\in\mathbb{R}}$.
- Full de Sitter: have worked out expressions for the projectors $P_{[-\Omega,\Omega]}(-\Box_k)$ but still deriving sampling formulas
- Techniques used so far:
  - complex analysis
  - operator theory: self-adjoint extensions of symmetric operators, eigenfunction expansion theory of Sturm-Liouville differential operators, spectral theory

Outlook

- Would like to develop full understanding of reconstruction properties of $B(M, \Omega)$ for physically relevant spacetimes: Power law, Schwarzschild, etc.
- spatial sampling formulas for fixed temporal frequencies, higher dimensions
- use techniques from communication engineering- e.g. oversampling to improve convergence rates of sampling formulas
- more general/abstract results for sampling on pseudo-Riemannian manifolds