

Geometric discretisation of GR

AMS Meeting New Orleans, January 2007



Motivation



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- invariance under arbitrary diffeomorphisms
 - any two points can be interchanged by a diffeo
 - points do not have any meaning



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- GR is a geometric theory
- invariance under arbitrary diffeomorphisms
 - any two points can be interchanged by a diffeo
 - points do not have any meaning
- Only relations between several points carry information



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- geometry is determined by relations between points and higher dimensional objects
 - „distance“ between two points



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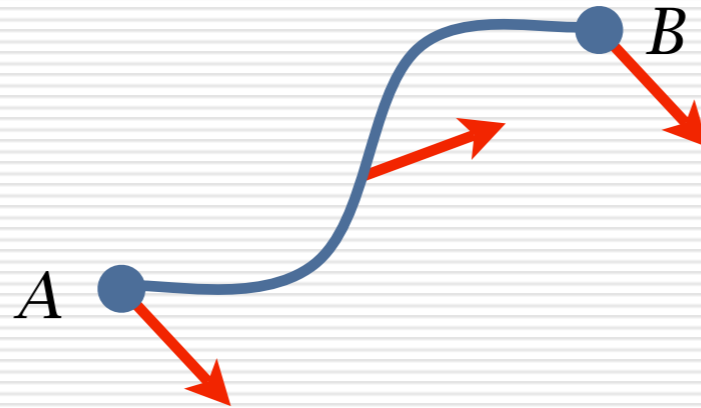
Motivation

- geometry is determined by relations between points and higher dimensional objects
 - parallel transport from A to B along a curve



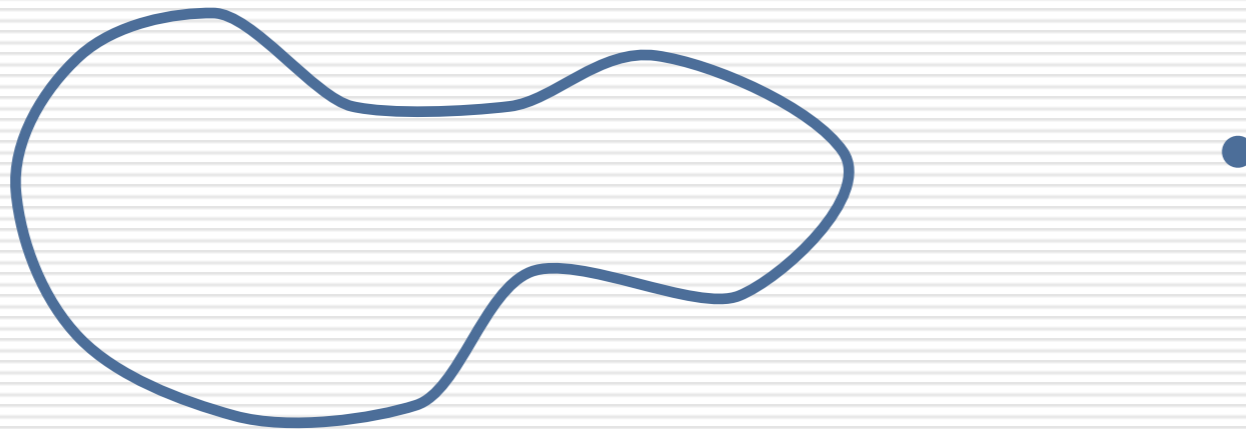
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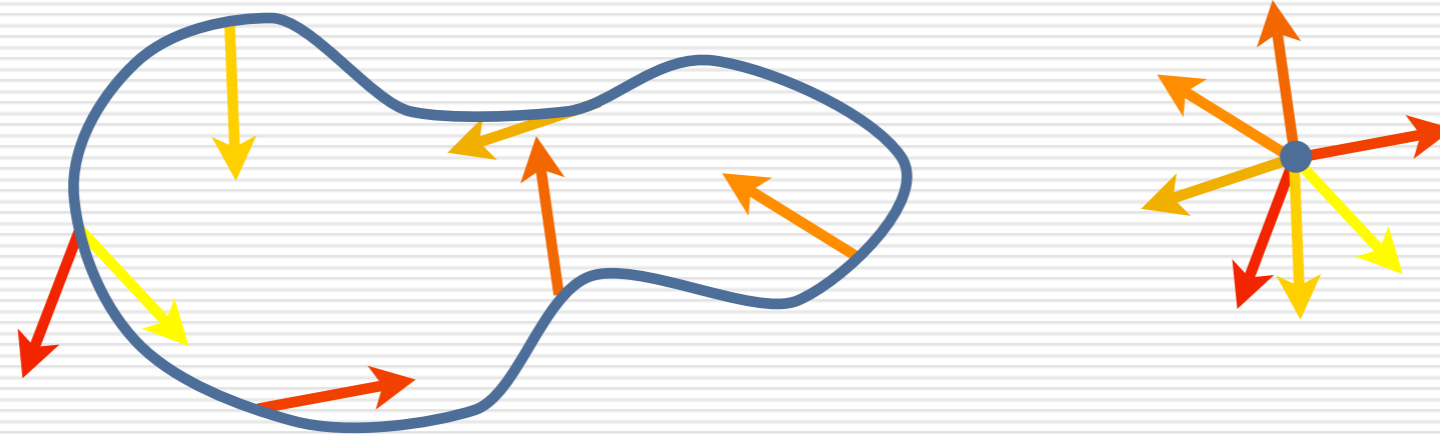
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 - holonomy around a closed path



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➔ Gauss curvature of a surface



Consequence

- localise objects not only on points
 - FD: tensor components as grid functions
- but also on lines, surfaces, volumes according to their meaning



Maxwell theory

1-form: electric field

$$E = E_x dx + E_y dy + E_z dz$$



Maxwell theory

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line = 1-dim sub-mf



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$$E : \begin{array}{c} B \\ | \\ L \\ | \\ A \end{array} \mapsto \int_L E$$

work done on unit charge along L
(voltage between A and B)



Maxwell theory

2-form: magnetic induction

$$B = B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy$$

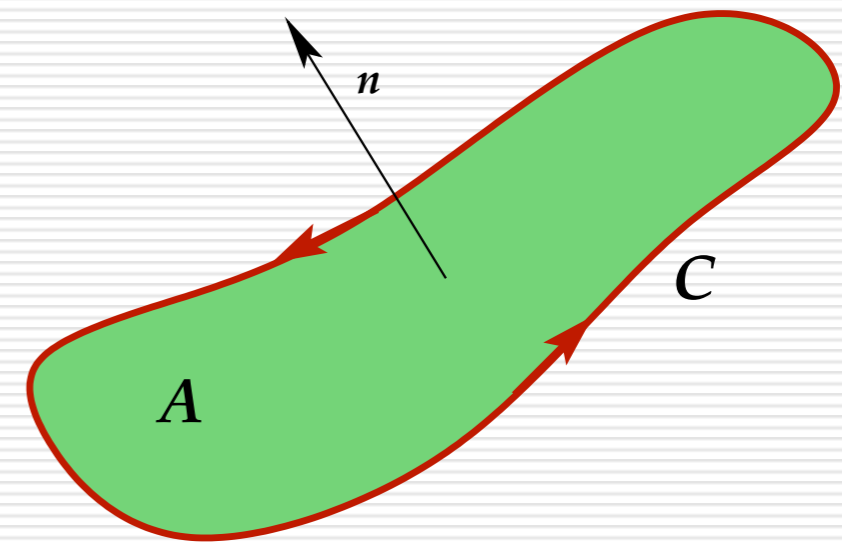


Maxwell theory

2-form: magnetic induction

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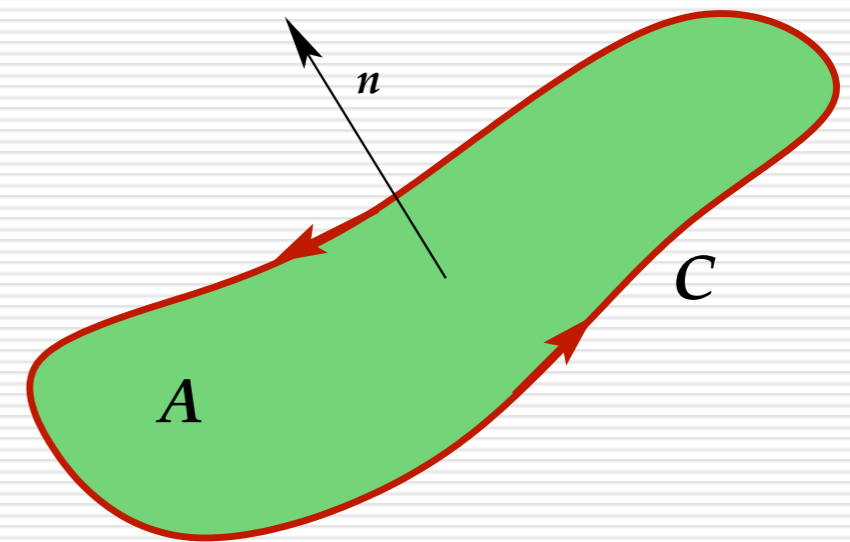


Maxwell theory

2-form: magnetic induction

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$$B = B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy$$



$$B : \text{[Diagram of surface A]} \mapsto \int_A B$$

magnetic flux through surface A



Maxwell theory

Maxwell equations:

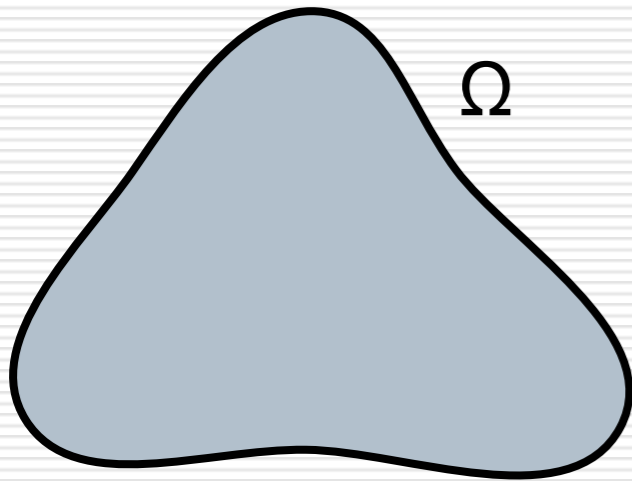
$$\dot{B} = \text{rot}E \iff \frac{d}{dt} \int_A B = \int_{\partial A} E$$

$$\dot{E} = -\text{rot}B \iff \frac{d}{dt} \int_A E = - \int_{\partial A} B$$



Discretisation of Maxwell

$$\dot{B} = \text{rot} E$$

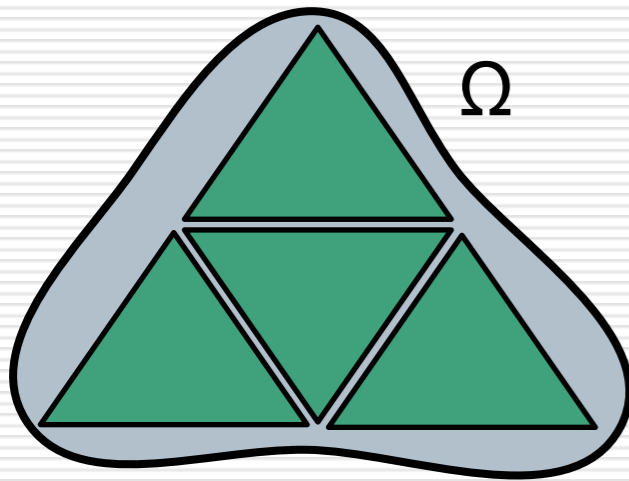


$$\frac{d}{dt} \int_{\Omega} B = \int_{\partial\Omega} E$$



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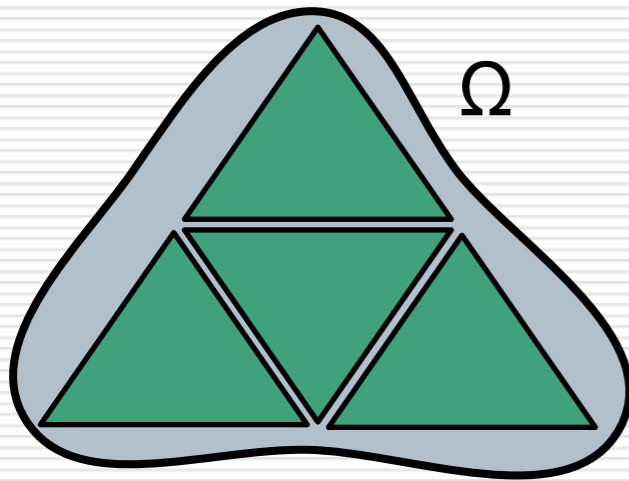


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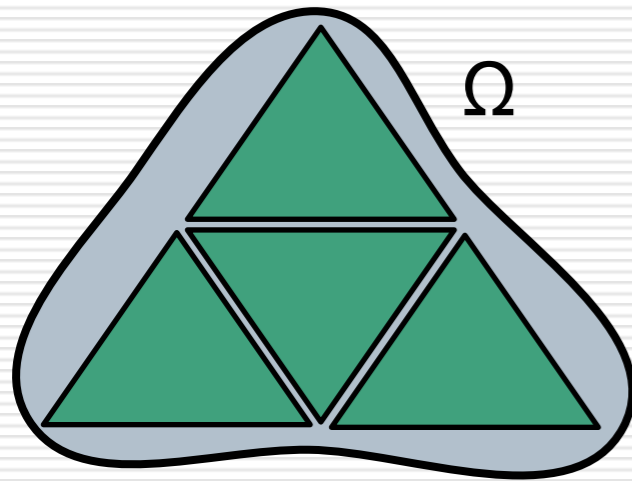


$$\frac{d}{dt} \sum_i \int_{\Delta_i} B = \sum_i \int_{\partial \Delta_i} E$$

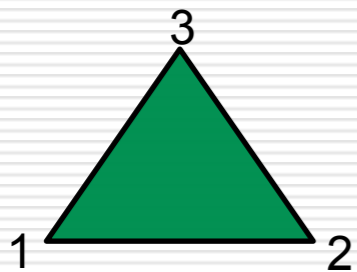


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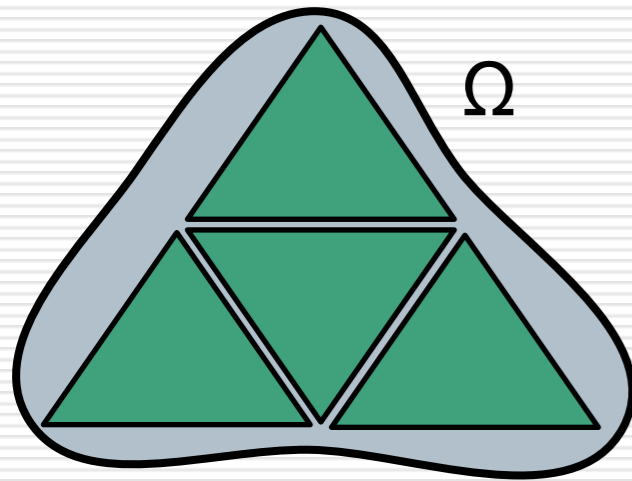


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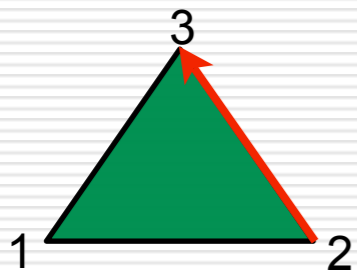


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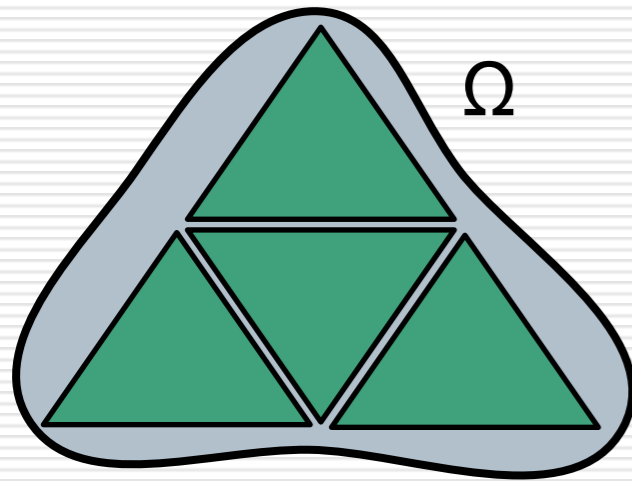


$$b = \int_{\Delta} B \quad e_1 = \int_2^3 E$$

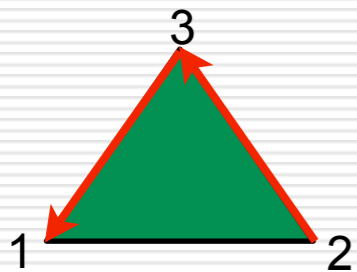


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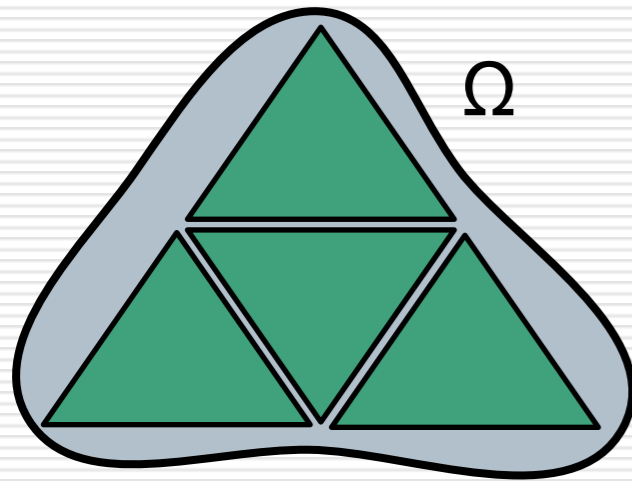


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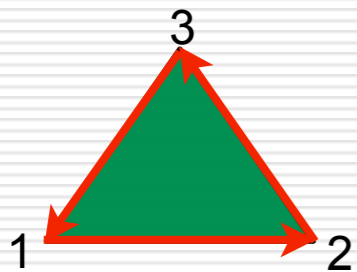


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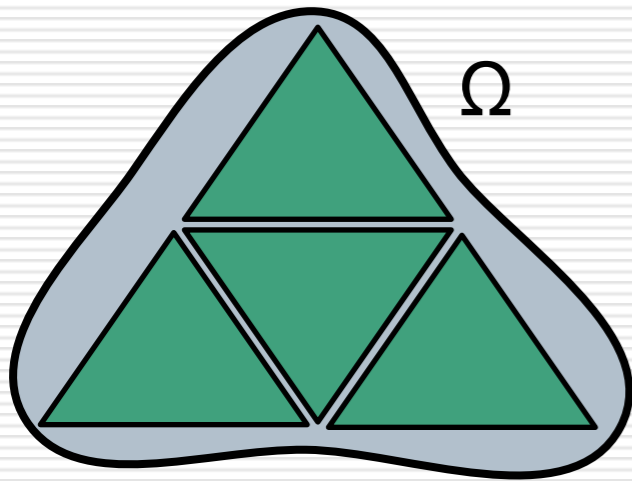


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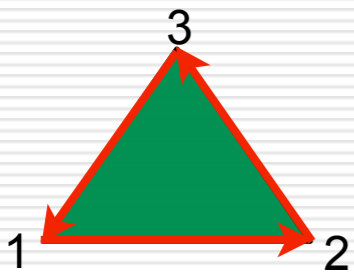


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$$\dot{b} = e_1 + e_2 + e_3$$



GR with differential forms

tetrad

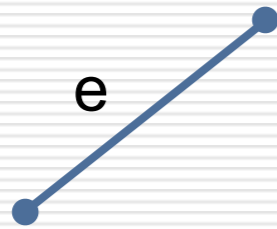
$$(\theta^0, \theta^1, \theta^2, \theta^3)$$



GR with differential forms

tetrad

θ^i



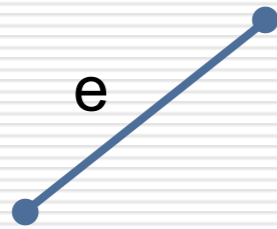
$$l^2 = \eta_{ik} \theta^i[e] \theta^k[e]$$



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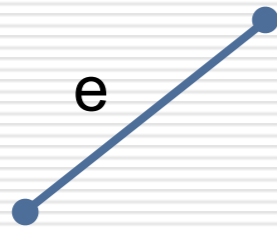
connection



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$$\theta^i$$



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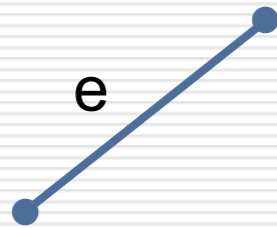
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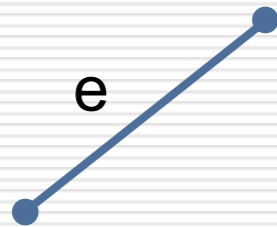
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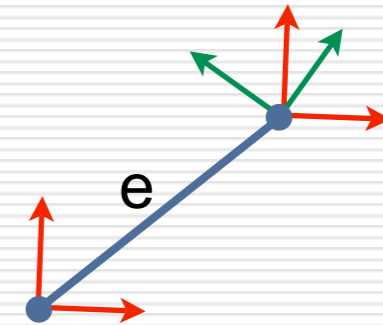
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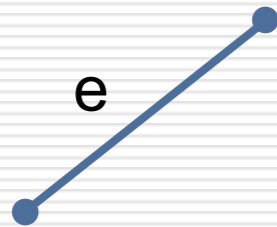
$\exp(\omega)$: holonomy along e



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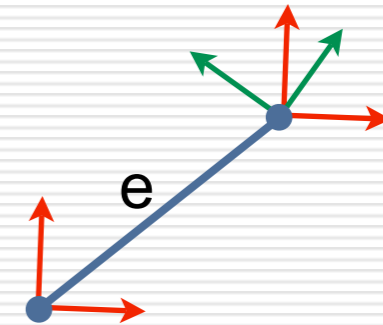
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related by the torsion free condition

$$d\theta^i + \omega^i_k \wedge \theta^k = 0$$



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Einstein equation

$$\mathbf{dL}_i = \mathbf{S}_i$$



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- Penrose inequality (?)



Results



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Coworkers

- * Ronny Richter
- * Marlene Vogel
- * Christian Apeltauer



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 - Geometric discretisation by local domains of dependence (\sim lightlike coordinates)
 - found order of convergence

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Coworkers

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J. Frauendiener, CQG 23, S369-S385, 2006

JF, R. Richter, M. Vogel, CQG 24, 433-453, 2007

