

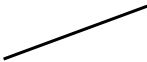
¶ 1. Most of us can easily recognize sequences of positive integers like

- (a) $\{1, 2, 4, 6, 8, \dots\}$ (the doubling sequence)
- (b) $\{1, 4, 9, 16, 25, \dots\}$ (Square Numbers))
- (c) $\{2, 3, 5, 7, 11, 13, \dots\}$ prime numbers
- (d) $\{1, 1, 2, 3, 5, 8, 13, \dots\}$ (Fibonacci Numbers)
- (e) $\{1, 3, 6, 10, 15, \dots\}$ (Triangular numbers)

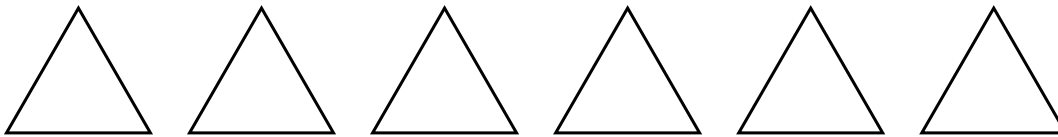
Often you can devise a procedure that generates the sequence, but that, if possible, can take quite some time. If you ever run into a sequence of integers, you can do no better than visit *The On-Line Encyclopedia of Integer Sequences* (www.research.att.com/njas/sequences/) developed by N.J.A. Sloane at AT&T Labs.

¶ 2. This problem was considered by Euler in the eighteenth century. In how many ways can a fixed convex polygon of $n + 2$ sides (or vertices) be divided into triangles by drawing diagonals that do not intersect?

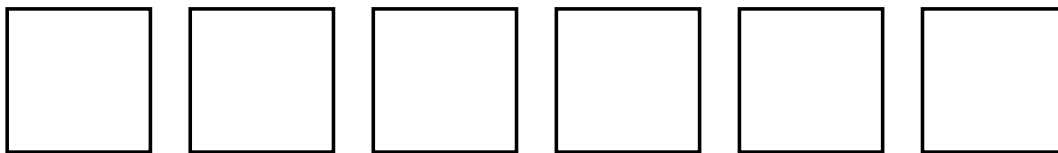
- (a) $0 + 2$ sides; this is just a segment joining two points.



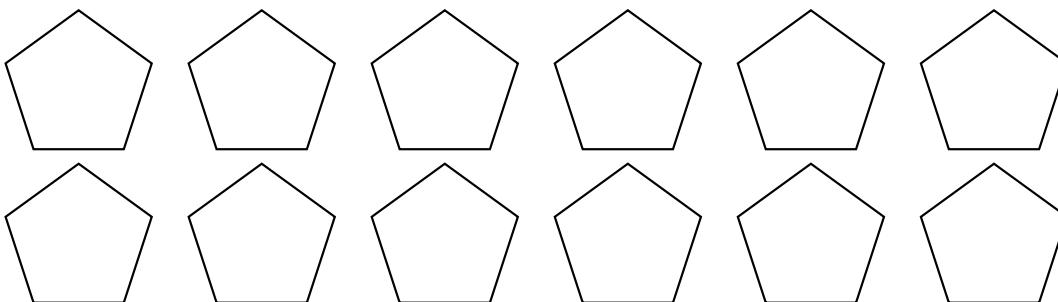
- (b) $1 + 2$ sides; a triangle



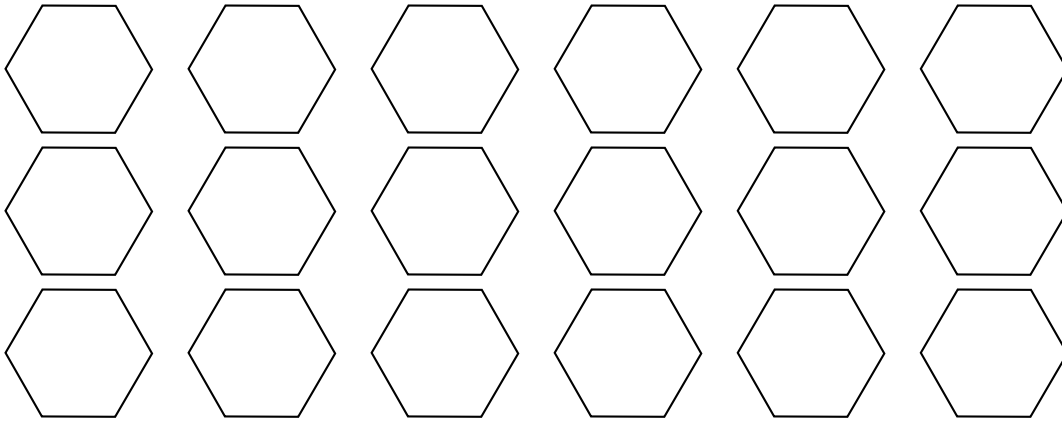
- (c) $2 + 2$ sides; a square.



- (d) $3 + 2$ sides; a pentagon.



(e) $3 + 3$ sides; a hexagon.



¶ 3. How many rooted plane trivalent trees are there with n internal nodes? (“Plane” means that it is drawn on the plane. “Trivalent” means that three branches meet at each node or bifurcation.)

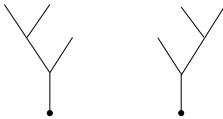
(a) $n = 0$ bifurcations



(b) $n = 1$ bifurcation



(c) $n = 2$ bifurcations



(d) $n = 3$ bifurcations

(e) $n = 4$ bifurcations

(f) $n = 5$ bifurcations

¶ 4. Evaluating laddered exponentials. How many values can you expect from a n -fold exponential?
For example:

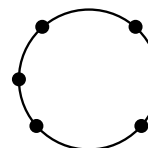
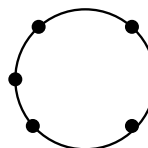
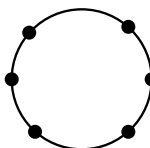
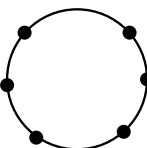
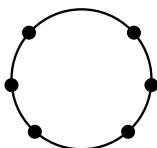
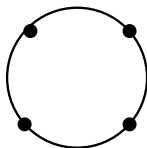
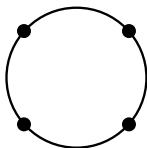
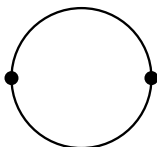
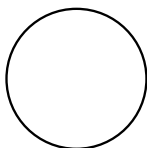
(a) 2

(b) $3^2 = 9$

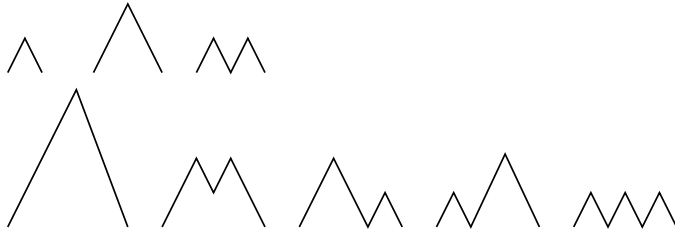
(c) $(4^3)^2 = 4^6$; $4^{(3^2)} = 4^9$

(d) $((5^4)^3)^2 = 5^{24}$;

¶ 5. There are n pairs of people sitting around a round table. In how many non-crossing handshakes are possible?



¶ 6. How many mountain landscapes can you draw with n upstrokes and n downstrokes?



¶ 7. Homer Simpson leaves Moe's Tavern with one too many. He staggers out of the door straight ahead, one step at a time, but sometimes he takes a step forward, sometimes backwards, randomly. So confused he is, that after an even number of steps he may end up back at Moe's. In how many ways can he take $2n$ steps that will return him to Moe's door?

¶ 8. The sequences that you have obtained are all the same sequence:

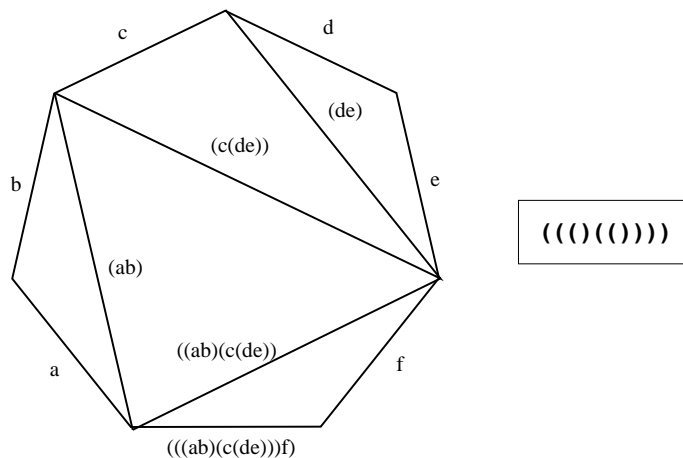
$$1, 1, 2, 5, 14, 42, \dots$$

The best way to see that, without actually computing the sequence, is to translate any one of these enumeration problems into any other enumeration problem.

The numbers in this sequence are called Catalan Numbers, named for the Belgian mathematician Eugene Catalan, who in 1838 solved the following problem: How many ways are there of arranging n pairs of parentheses?

- (a) $n = 0$:
- (b) $n = 1$: ()
- (c) $n = 2$: (()), ()().
- (d) $n = 3$: ((())), (())(), ...
- (e) $n = 4$

¶ 9. The following figure explains the correspondence between triangulations of polygons and parenthesis:



¶ 10. There is a nice recurrent formula for computing the next term. Write all the terms obtained so far from left to right on a row, and from right to left on a second row. Then multiply each number on the top row by the one below it, and add all the products; the result is the next number in the sequence. For example, to find the number following 1, 1, 2, 5, 14, 42, you write

$$\begin{array}{cccccc} & 1 & 1 & 2 & 5 & 14 & 42 \\ \times & 42 & 14 & 5 & 2 & 1 & 1 \\ \hline & & & & & & \end{array} =$$

¶ 11. Surely you are familiar with Pascal's Triangle

$$\begin{array}{ccccccc} & & & & 1 & & & & \\ & & & & 1 & & 1 & & \\ & & & 1 & & 2 & & 1 & \\ & & 1 & & 3 & & 3 & & 1 \\ & 1 & & & & & & & \end{array}$$

Look at the middle numbers of each other row in the triangle:

$$1, 2, 6, 70, 252, 924, \dots$$

These numbers can be divided by the numbers

$$1, 2, 3, 4, 5, 6, \dots$$

to give the sequence of Catalan numbers.

$$1, 1, 2, 5, 14, \dots$$

If you know the combinatorial form for the numbers in Pascal's triangle: $\binom{n}{m}$ for the $(m + 1)$ th number in the $(n + 1)$ th row (we take $n, m = 0, 1, 2, 3 \dots$), then the Catalan numbers are:

Literature

[1] John Conway and Richard Guy, *The Book of Numbers*, Springer, New York, 1996.
 [2] Martin Gardner, *Time Travel and other Mathematical Bewilderments*, W. H. Freeman and Co., San Francisco, 1988.