¶ **1.** We group sets into classes: Two sets are in the same class if there is a one-one correspondence between them. Any set that is either finite, or in the class of a set of natural numbers of the form $\{1, 2, \cdots, n\}$ is called a finite set. Otherwise it is called an infinite set. Thus, if a set is infinite, then for any natural number n, if we remove n elements from the set, there will be elements left over; in fact, infinitely many.
Try this: from an infinite set, remove just one element. Is the set that results finite or infinite?

¶ **2.** What all the problems that we have discussed last week reveal is that infinite sets have the peculiar property that they can be put into one-one correspondence with parts of themselves.
A set A is called a subset of a set B if every member of A is also a member of B. For example, if B is the set of all triangles and A is the set of all equilateral triangles then A is a subset of B; it is in fact a proper subset because not every triangle is an equilateral triangle.
What if B is the set of all triangles and A is the set of all triangles with 4 sides?

¶ **3.** The set $P = \{1, 2, 3, \ldots\}$ can be put into one-one correspondence with the set $Q = \{2, 3, \ldots\}$. This was the key to dealing with the first Hilbert's Hotel problem.
Many years ago, Galileo observed that the positive integers can be put into one-one correspondence with the squares

$$\begin{array}{ccccccc} 1 & 4 & 9 & 16 & \cdots & n^2 & \cdots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & & \updownarrow & \\ 1 & 2 & 3 & 4 & \cdots & n & \cdots \end{array}$$

This seems to contradict the ancient axiom that the whole is greater than any of its parts.

¶ **4.** Two sets are of the same size if they can be put into one-one correspondence. If A is a proper subset of B, then in the obvious sense, B is greater than A, because B contains all the members of A and some more. If furthermore B is finite, then A actually less members than B in the numerical sense. We may this be tempted to say that A has smaller size than B if A can be put into one-one correspondence with a proper subset of B. But if B is infinite, this may not be adequate. For example, if E is the set of all even numbers and O is the set of all odd numbers, then we can put O and E into one-one correspondence, and we can put O into a one-one correspondence with a subset of E, and we can put E in a one-one correspondence with a proper subset of O.

¶ **5.** Cantor introduced the cardinal numbers and resolved these problems in the following way. The cardinal number of a set A is denoted by the symbol $o(A)$. Two sets have the same cardinal numbers if they can be put into one-one correspondence. The cardinal number of a set is less than or equal to the cardinal number of a second set is there is a one-one correspondence between the first set and a subset of the second set. In symbols, we say that $o(A) \leq o(B)$ is there is a one-one correspondence between A and a subset of B.
We say that $o(A) < o(B)$ if $o(A) \leq o(B)$ and $o(A) \neq o(B)$. In words, the cardinal number of the set A is strictly smaller than the cardinal number of the set B if: (1) A can be put into one-one correspondence with a subset of B; and (2) A cannot be put into a one-one correspondence with all of B.

¶ **6.** The cardinal number of a finite set is denoted by the natural number that indicates the number of elements in that sets. Thus $o(\emptyset) = 0$, $o(\{\emptyset\}) = 1$, $o(\{\emptyset\}, \{\emptyset\}\}) = 2$, and so on. The cardinal number of the set of natural numbers is denoted by $\aleph_0$ (aleph-not, $\aleph$ is a letter in the Jewish alphabet).
We have seen sets with cardinal numbers $0, 1, 2, \ldots, \aleph_0$. Are any others?

¶ **7.** Motivated by his work on trigonometric series, Cantor arrived at the question of whether two infinite sets must be of the same size, that is, have the same cardinal number. For example, which equation has more solutions $\sin x = 0$ or $\sin(e^{\tan x}) = 0$?
Cantor initially conjectured that the answer was positive, but after a twelve years of research on the problem he discovered that the answer was negative: there are infinite sets of cardinality bigger than $\aleph_0$!
A set is called denumerably infinite, or just denumerable, if it can be put into one-one correspondence with the set of natural numbers. So the question that Cantor analyzed was: are all infinite sets denumerable? He considered infinite sets that appear to be too large to be denumerable, but he could enumerate them after all.
To enumerate a set is to establish a one-one correspondence with the set of all whole numbers. For example, probe that the set of all integers $\cdots, -2, -1, 0, 1, 2, \cdots$ is countable.

¶ **8.** Make a set whose members are pairs of whole numbers. Is it possible to enumerate this set?

¶ **9.** Now make a set whose members are ordered pairs of whole numbers. Describe an strategy to enumerate this set.

¶ **10.** How could you enumerate the set of finite subsets of whole numbers?

¶ **11.** How could you enumerate the set of all positive fractions?

¶ **12.** Here is another method of solving the above problem, proposed by Charles Pierce, to enumerate the positive fractions. Start with $0 = \frac{0}{1}$ and $\frac{1}{0}$. Sum the two numerators and then the two denominators to get a new fraction in between

$$\frac{0}{1} \quad \frac{0+1}{1+0} \quad \frac{1}{0}$$

Repeat this procedure with each pair of adjacent fractions to obtain two new fractions that go in between the above:

$$\frac{0}{1} \quad \frac{1}{2} \quad \frac{1}{1} \quad \frac{2}{1} \quad \frac{1}{0}$$

and on:

$$\frac{0}{1} \quad \frac{1}{3} \quad \frac{1}{2} \quad \frac{2}{3} \quad \frac{1}{1} \quad \frac{3}{2} \quad \frac{2}{1} \quad \frac{3}{1} \quad \frac{1}{0}$$

and on

$$\frac{0}{1} \qquad \frac{1}{3} \qquad \frac{1}{2} \qquad \frac{2}{3} \qquad \frac{1}{1} \qquad \frac{3}{2} \qquad \frac{2}{1} \qquad \frac{3}{1} \qquad \frac{1}{0}$$

The series thus obtained is related to the so called Farey series. It has many peculiar properties. Each fraction will appear once and only once, and always in its simple rational form. At each new step, the digits above the fraction lines, taken in order from left to right begin by repeating the digits of the previous step. And at each new step, the digits below the fraction lines are the same as those above the lines but in reverse order. As a consequence, any two fraction that are equidistant from the central fraction $\frac{1}{1}$ are reciprocals of each other. Also, adjacent fractions $\frac{a}{c}$ and $\frac{b}{d}$ satisfy $ad - bc = -1$

¶ **13.** We can do arithmetic with cardinal numbers. For example, to add 2 and 5, we find two sets, one set with 2 members and a second set with 5 members. If we combine the members of those two sets and form a larger set whose members are those individuals that are either in the first set or in the second set, the we have a set with 7 elements.
In general, to add cardinals a and b, we find sets A and B with $o(A) = a$ and $o(B) = b$, and then we set $a + b$ as the cardinal of the combination of A and B: the union $A \cup B$.

¶ **14.** We can also multiply cardinal numbers. If we have a set with 3 member and a second set with 7 members, we make a set consisting of pairs, one from each set. How many member does this set have?
In general, if we have cardinal numbers a and b, we find sets A and B with $o(A) = a$ and $o(B) = b$, and the set $a \cdot b$ to be the cardinal of the sets of all pairs (x, y) where x is in A and y is in B.

¶ **15.** What is the result of the following operations?

(a) $\aleph_0 + 1$

(b) $\aleph_0 + \aleph_0$

(c) $3 \cdot \aleph_0$

(d) $\aleph_0 \cdot \aleph_0$

¶ **16.** (a) The set of points of any two segments of different lengths can be put into one-one correspondence. How?

(b) The set of points in the line and the set of points on the interval (0, 1) can be put into one-one correspondence. How?

¶ **17.** A lottery runs tickets numbered 00000 to 99999; five digits for a total 10^5 tickets. We could imagine lottery tickets with infinitely many digits $d_1 d_2 d_2 \cdots$. By analogy, we may agree that the cardinal number of such set of lottery tickets is $10^{\aleph_0}$.
Cantor demonstrated that the set of all such tickets is not denumerable, thus establishing that $\aleph_0 < 10^{\aleph_0}$.
This is how he accomplished that. It uses an argument by contradiction. Assume that we could effectively enumerate the set of infinite lottery tickets, thus establishing a one-one correspondence:

$$\begin{array}{rcl} 1 & \leftrightarrow & a_1 a_2 a_3 \cdots \\ 2 & \leftrightarrow & b_1 b_2 b_3 \cdots \\ 3 & \leftrightarrow & c_1 c_2 c_3 \cdots \\ 4 & \leftrightarrow & d_1 d_2 d_3 \cdots \\ 5 & \leftrightarrow & e_1 e_2 e_3 \cdots \\ \cdots & & \cdots\cdots \\ n & \leftrightarrow & r_1 r_2 r_3 \cdots \\ \cdots & & \cdots\cdots \end{array}$$

The you can write down an infinite ticket number that is not in the list. How do you do that? You chose a first digit x_1 distinct from a_1, a second digit x_2 distinct from b_2, a third digit x_3 distinct from c_2, and so on. The resulting ticket is labeled $x_1 x_2 x_3 \cdots$, and it is definitely not in the above list. Do you see why?

¶ **18.** How many real number are there between 0 and 1? Lets call this number c (for *continuum*) Any such real number can be specified by an infinite decimal

$$0.abcd \cdots$$

in which there are $\aleph_0$ digits, each chosen from ten possibilities $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$. Because of the lottery ticket problem, there are $10^{\aleph_0}$ such decimal expansions. However, some of these infinite decimals really represent the same number: for example, $0.500000 \cdots = 0.499999 \cdots$. Each of this exceptions is in fact a rational number, and there are in fact $\aleph_0$ such exceptional numbers. Therefore:

$$10^{\aleph_0} = c + \aleph_0.$$

Now we can prove that $c + \alpha_0 = \aleph_0$. This is because any real number is either rational or irrational. If x denotes the number of irrational numbers, then $c = \aleph_0 + x$. Since $\aleph_0 + \aleph_0 = \aleph_0$, we deduce that

$$\aleph_0 + c = \aleph_0 + (\aleph_0 + x) = (\aleph_0 + \aleph_0) + x = \aleph_0 + x = c$$

Therefore, $10^{\aleph_0} = c$. If we have used base 2 instead of base 10 for representing real numbers as infinite decimals, then we would have obtained the standard equation

$$\boxed{2^{\aleph_0} = c}$$

¶ **19.** Establish the following equations by finding appropriate one-one correspondences:

(a) For any $n = 0, 1, 2, 3, \cdots$

$\aleph_0 = n + \aleph_0 = \aleph_0 + \aleph_0$

$c = n + c = \aleph_0 + c = c + c$

(b) For any $n = 1, 2, 3, \cdots$

$\aleph_0 = n \times \aleph_0 = \aleph_0 \times \aleph_0$

$c = n \times c = \aleph_0 \times c = c \times c$

(c) For any $n = 2, 3, \cdots$

$\aleph_0 = \aleph_0^n$

$c = c^n = n_0^{\aleph} = \aleph_0^{\aleph_0} = c^{\aleph_0}$

¶ **20.** Thus Cantor showed that $\aleph_0 < 2_0^{\aleph}$: the sets of subsets of natural number has cardinality strictly bigger than that of the set of all natural numbers. Indeed, you can identify a subset of the set of natural number with an infinite sequence of 0's and 1's: The *n*th term of this sequence is 0 if n is not in the set, and 1 if n is in the set.
Cantor diagonal argument in fact constructs infinitely many numbers of larger and larger size. If A is any set, the set of all subsets of A is denoted by $P(A)$. For example, if $A = \{\triangle, \square\}$, then $P(A)$ has 4 members:

$$\emptyset, \{\triangle\}, \{\square\}, \{\triangle, \square\}$$

¶ **21.** For any set A, there is always a one-one correspondence of A into a subset of $P(A)$: it takes any a to the one element subset $\{a\}$, which is a member of $P(A)$.
However no such one-one correspondence can be obtained between A and $P(A)$. It there were such matching $a \leftrightarrow S_a$, then we could divide the elements of A into two classes, red and blue, as follows:

(a) The element a is blue if the element a is assigned to a subset S_a that includes a as one of its members.

(b) The element a is red if the set S_a does not include a as one of its members.

The red elements form a subset, R, of the original set A, which is matched to a unique member of A, say x, so that $S_x = R$. What color is x? If x is blue, then x is a member of $R = S_x$, which is a contradiction because R consists of all red members. If x is red, then x is in $R = S_x$ by the definition of R; this would make x to be blue, again a contradiction.
Therefore, for any set A the following inequality of ordinal numbers holds true:

$$\boxed{o(A) < o(P(A)) = 2^{o(A)}}$$

¶ **22.** (a) How many circles (of arbitrary radius) can be placed in the plane in such a way that no two circles overlap and no circle contains another circle in its interior?

(b) How many circles of radius 1 can be placed in the plane in such a way that no two overlap?

(c) How many figure 8 can be place in the plane in such a way that no two overlap? The figure 8 can be of any size and have any position.

¶ **23.** How many replicas of a given figure (for example, a letter of the alphabet) can be place on the plane with no overlap and no intersection? Replicas must be geometrically similar to the given model, but can vary in size and position.

¶ 24. Suppose that you could enumerate the real numbers in a sequence $x_1, x_2, \cdots$. Enclose the number x_1 in an interval of length $1/10$, the number a_2 in an interval of length $1/10^2$, and so on. If all the real numbers are on the list $x_1, x_2, \cdots$, then the sum of the lenght of those interval should be infinite. But, What is the sum of the lengths of those intervals? (Hint: their lengths form a geometric progression).

¶ 25. Playing with sets requires a lot of care. A branch of mathematics, Set Theory, establishes precisely the axioms a rules that are to be followed for deducing theorems. It is easy to arrive a bewildering paradoxes of no care is taken in using the concept of "set." One such paradox is called the Russell Paradox, and goes as follows.
Most sets do not contain themselves as elements. For example, the set formed by the integers $0, 1, -1$, that is $\{0, 1, -1\}$, is not a member of itself. The set formed by all students in this classroom is not a member of itself (because it is not a student but a collection of students); the set of all integers is not an integers, hence it cannot be a member of itself. We call all such set "ordinary." Most sets are ordinary, but it may be possible to find sets that are not ordinary. For example, the set S defined as follows: "S contains all the sets that are definable by an English phrase of no more than twenty words" could be considered to contain itself as a member. We call such sets "extraordinary." Now we confine our attention to the set of all ordinary sets and call this set C. Each element of the set C is itself a set; in fact it is an ordinary set. A legitimate question thus arises: is C itself an ordinary set or an extraordinary set? Since any set is either ordinary or extraordinary (but not both), C must be one or the other. If C is ordinary, then C must be among the members of C, since C was the set of all ordinary sets. This is a contradiction because no ordinary set can be a member of itself. Thus C must be extraordinary. But then C contains an extraordinary member, namely itself, which contradicts the definition whereby C was to contain ordinary sets only. Thus we see that the assumption of the existence of C leads to a contradiction.

¶ 26. Bertrand Russell used to put his paradox in the following pictorial fashion (thus the Barber Paradox).
Suppose there is a town with just one male barber; and that every man in the town keeps himself clean-shaven: some by shaving themselves, some by attending the barber. It seems reasonable to imagine that the barber obeys the following rule: He shaves all and only those men in town who do not shave themselves.
Under this scenario, we can ask the following question: Does the barber shave himself?
Asking this, however, we discover that the situation presented is in fact impossible:

(a) If the barber does not shave himself, he must abide by the rule and shave himself.

(b) If he does shave himself, according to the rule he will not shave himself.

¶ 27. Cantor guessed hat there is no number strictly between $\aleph_0$ and $c = 2^{\aleph}_0$, and spent a long period of his life trying to prove that. It became known as *the continuum hypothesis*. Thanks to the efforts of Kurt Gödel and Paul Cohen that Cantor's guess can never be proved nor disproved from other axioms of Set Theory.

Literature

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