¶ **1.** You probably still remember that time in your early childhood when you realized that there were no limits to counting.

¶ **2.** The word "infinite" is an adjective, and we must first agree upon the sort of things to which that adjective is applicable. What sort of things are finite or infinite? In the mathematical use of the term, it applies to sets, well defined collections of things.

¶ **3.** How do we determine which sets are finite and which sets are infinite? The key concept lies in that of correspondence, an abstraction of counting. When we begin to learn how to count as children, we often use our fingers to label the members of collections: how many toys I have? How many fingers do I need to label them? How many toys do you have?... We compare collections by assigning one finger to each member of the collection.

¶ **4.** A finite set is any set whose elements can be put into one-one correspondence with a set of the form $\{0, 1, 2, \cdots, n-1\}$. Such set is finite and contains n elements.

¶ **5.** The set of natural numbers consists of all positive whole numbers $1, 2, \cdots$ and 0.

¶ **6.** To each finite set we assign a natural number, called its cardinal number, in a way that two finite sets can be put into one-one correspondence if and only if they have the same cardinal number. The number 0 appears as a cardinal number because it is the cardinal of the empty set.

¶ **7.** The empty set is a very abstract concept. It is a set that contains no members. In mathematics it is denoted by the symbol $\emptyset$ or $\{\}$. It happens that there is exactly one empty set. This is because one of the axioms of set theory (the axiom of extensionality) states that two sets are equal if and only if they contains the same members.

¶ **8.** The empty set if not a set that contains nothing. It is for example the set of all real numbers a such that $a < a$. It is the set of all real numbers a such that $a^2 < 0$. It is the set of all triangles with four sides. If A is any set, then $\{x \in A \mid x \neq x\}$ is the empty set.
The empty set is a mathematical necessity; without it many exceptions would have to be made and mathematical language would be truly cumbersome.

¶ **9.** You have a set of tennis balls labeled $1, 2, 3, 4, \cdots$, one for each positive whole number. You are going to put them into two bins, A and B according to the following moves. On the first move, you place balls 1 and 2 into Bin A, and then move ball 1 to Bin B. On the second move, you place balls 3 and 4 into Bin A and then move ball 2 (which is in Bin A) into Bin B, and so on. How many balls are in Bin A when you are done?

¶ **10.** We group sets into classes: Two sets are in the same class if there is a one-one correspondence between them. Any set that is either finite, or in the class of a set of natural numbers of the form $\{1, 2, \cdots, n\}$ is called a finite set. Otherwise it is called an infinite set. Thus, if a set is infinite, then for any natural number n, if we remove n elements from the set, there will be elements left over; in fact, infinitely many.
Try this: from an infinite set, remove just one element. Is the set that results finite or infinite?

¶ **11.** Infinite sets are even trickier: it is possible to remove an infinite set from an infinite set in such a way that the remainder is infinite. A good illustration of this property is the famous story of Hilbert's Hotel.
Suppose that a hotel has any finite number of rooms, say 100. Suppose that the hotel is full, and each room has exactly one guest. If a new person arrives wanting a room for the night and none of the current occupants wants to share a room, then it will not be possible to accommodate the new arrival: it is not possible to put 100 rooms into one-one correspondence with 101 persons.
However, if the hotel has an infinite number of rooms, the situation is quite different. Hilbert's Hotel has infinitely many rooms, one for each whole number, labeled Room 1, Room 2, ..., and so on. Assume that all rooms are occupied and that a new person arrives. No guest is willing to share a room, but the manager of the Hilbert's Hotel (Hilbert himself) manages to accommodate the new arrival. How did he do it?

¶ **12.** More perplexing things can occur. Suppose that not just one new person arrives, but an infinite number of prospective guests arrive: $P_1, P_2, \ldots$. Still, nobody is willing to share a room. What does Hilbert do to accommodate them?

¶ **13.** Even more. Hilbert's management company owns a chain of hotels, $H1$, $H2$, $H3$, …, one for each whole number, each with infinitely many rooms, one for each whole number. In order to cut costs, one good day, the management company decides to shut down all the hotels but one. This means that all the occupants of all the hotels must be accommodated into just one of the hotels. How can this be accomplished?

¶ **14.** What all these problems reveal is that infinite sets have the peculiar property that they can be put into 0ne-one correspondence with parts of themselves.
A set A is called a subset of a set B if every member of A is also a member of B. For example, if B is the set of all triangles and A is the set of all equilateral triangles then A is a subset of B; it is in fact a proper subset because not every triangle is an equilateral triangle.
What if B is the set of all triangles and A is the set of all triangles with 4 sides?

¶ **15.** The set $P = \{1, 2, 3, \ldots\}$ can be put into one-one correspondence with the set $Q = \{2, 3, \ldots\}$. This was the key to dealing with the first Hilbert's Hotel problem.
Many years ago, Galileo observed that the positive integers can be put into one-one correspondence with the squares

$$\begin{array}{cccccccc} 1 & 4 & 9 & 16 & \cdots & n^2 & \cdots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & & \updownarrow & \\ 1 & 2 & 3 & 4 & \cdots & n & \cdots \end{array}$$

This seems to contradict the ancient axiom that the whole is greater than any of its parts.

¶ **16.** Two sets are of the same size if they can be put into one-one correspondence. If A is a proper subset of B, then in the obvious sense, B is greater than A, because B contains all the members of A and some more. If furthermore B is finite, then A actually less members than B in the numerical sense. We may this be tempted to say that A has smaller size than B if A can be put into one-one correspondence with a proper subset of B. But if B is infinite, this may not be adequate. For example, if E is the set of all even numbers and O is the set of all odd numbers, then we can put O and E into one-one correspondence, and we can put O into a one-one correspondence with a subset of E, and we can put E in a one-one correspondence with a proper subset of O.

¶ **17.** Cantor introduced the cardinal numbers and resolved this problems in the following way. The cardinal number of a set A is denoted by the symbol $o(A)$. Two sets have the same cardinal numbers if they can be put into one-one correspondence. The cardinal number of a set is less than or equal to the cardinal number of a second set is there is a one-one correspondence between the first set and a subset of the second set. In symbols, we say that $o(A) \leq o(B)$ is there is a one-one correspondence $A \leftrightarrow B$.
We say that $o(A) < o(B)$ if $o(A) \leq o(B)$ and $o(A) \neq o(B)$. In words, the cardinal number of the set A is strictly smaller than the cardinal number of the set B if: (1) A can be put into one-one correspondence with a subset of B; and (2) A cannot be put into a one-one correspondence with all of B.

¶ **18.** The cardinal number of a finite set is denoted by the natural number that indicates the number of elements in that sets. Thus $o(\emptyset) = 0$, $o(\{\emptyset\}) = 1$, $o(\{\emptyset\}, \{\emptyset\}\}) = 2$, and so on. The cardinal number of the set of natural numbers is denoted by $\aleph_0$ (aleph-not, $\aleph$ is a letter in the Jewish alphabet).
We have seen sets with cardinal numbers 0, 1, 2, …, $\aleph_0$. Are any others?

¶ **19.** Motivated by his work on trigonometric series, Cantor arrived at the question of whether two infinite sets must be of the same size, that is, have the same cardinal number. He initially conjectured that the answer was positive, but after a twelve years of research on the problem he discovered that the answer was negative: there are infinite sets of cardinality bigger than $\aleph_0$!

¶ **20.** A set is called denumerably infinite, or just denumerable, if it can be put into one-one correspondence with the set of natural numbers. So the question that Cantor analyzed was: are all infinite sets denumerable? He considered infinite sets that appear to be too large to be denumerable, but he could enumerate them after all.
To enumerate a set is to establish a one-one correspondence with the set of all whole numbers. For example, probe that the set of all integers $\cdots, -2, -1, 0, 1, 2, \cdots$ is countable.

¶ **21.** Make a set whose members are pairs of whole numbers. Is it possible to enumerate this set?

¶ **22.** Now make a set whose members are ordered pairs of whole numbers. Describe an strategy to enumerate this set.

¶ **23.** How could you enumerate the set of all positive fractions?

¶ **24.** We can do arithmetic with cardinal numbers. For example, to add 2 and 5, we find two sets, one set with 2 members and a second set with 5 members. If we combine the members of those two sets and form a larger set whose members are those individuals that are either in the first set or in the second set, the we have a set with 7 elements. We can also multiply cardinal numbers. If we have a set with 3 member and a second set with 7 members, we make a set consisting of pairs, one from each set. How many memeber does this set have?

¶ **25.** What is the result of the following operations?

(a) $\aleph_0 + 1$

(b) $\aleph_0 + \aleph_0$

(c) $3 \cdot \aleph_0$

(d) $\aleph_0 \cdot \aleph_0$

¶ **26.** A lottery runs tickets numbered 00000 to 99999; five digits for a total 10^5 tickets. We could imagine lottery tickets with infinitely many digits $d_1 d_2 d_2 \cdots$. By analogy, we may agree that the cardinal number of such set of lottery tickets is $10^{\aleph_0}$.
Cantor demonstrated that the set of all such tickets is not denumerable, thus establishing that $\aleph_0 < 10^{\aleph_0}$.
This is how he accomplished that. It uses an argument by contradiction. Assume that we could effectively enumerate the set of infinite lottery tickets, thus establishing a one-one correspondence:

$$\begin{array}{ccl} 1 & \leftrightarrow & a_1 a_2 a_3 \cdots \\ 2 & \leftrightarrow & b_1 b_2 b_3 \cdots \\ 3 & \leftrightarrow & c_1 c_2 c_3 \cdots \\ 4 & \leftrightarrow & d_1 d_2 d_3 \cdots \\ 5 & \leftrightarrow & e_1 e_2 e_3 \cdots \\ \cdots & & \cdots\cdots \\ n & \leftrightarrow & r_1 r_2 r_3 \cdots \\ \cdots & & \cdots\cdots \end{array}$$

The you can write down an infinite ticket number that is not in the list. How do you do that? You chose a first digit x_1 distinct from a_1, a second digit x_2 distinct from b_2, a third digit x_3 distinct from c_2, and so on. The resulting ticket is labeled $x_1 x_2 x_3 \cdots$, and it is definitely not in the above list. Do you see why?

Literature

[1] Raymond Smullyan, *Satan, Cantor & Infinity*, Dover Publications, Inc. New York, 2009.