¶ **1.** You have fifteen seconds. Using standard math notation, English words, or both, write the biggest number you can think of in the box below. Be precise enough for any person with adecuate math skills to determine exactly what number you have written.

¶ **2.** We use exponential notation for large numbers. Thus there are about 300, 000, 000, 000, 000, 000, 000, 000, 000, 000, 000, 000, 000, 000, 000, 000, 000, 000, 000, 000, 000, 000, 000, 000, 000, 000, 000, 000, 000, 000, 000, 000 atoms in the observable universe, which we abbreviate by 3×10^{74}.
Neither the Egyptians nor the Romans could have written such number.
Today we have names for many large numbers: *million* for 10^6, *billion* (a thousand million) for 10^9, *vigintillion* for 10^{63}.
We also have *googol* for 10^{100}. The term was coined in 1938 by 9-year-old Milton Sirotta, nephew of Edward Kasner. Kasner then extended the term to the larger ”googolplex,” which is $10^{10^{100}}$.

¶ **3.** Archimedes was perhaps the first to devise a system to really write big numbers. In his treatise *Sand Reckoner* he write:

> " There are some, king Gelon, who think that the number of the sand is infinite in multitude; and I mean by the sand not only that which exists about Syracuse and the rest of Sicily but also that which is found in every region whether inhabited or uninhabited. Again there are some who, without regarding it as infinite, yet think that no number has been named which is great enough to exceed its magnitude. And it is clear that they who hold this view, if they imagined a mass made up of sand in other respects as large as the mass of the Earth, including in it all the seas and the hollows of the Earth filled up to a height equal to that of the highest of the mountains, would be many times further still from recognizing that any number could be expressed which exceeded the multitude of the sand so taken.
>
> But I will try to show you by means of geometrical proofs, which you will be able to follow, that, of the numbers named by me and given in the work which I sent to Zeuxippus, some exceed not only the number of the mass of sand equal in magnitude to the Earth filled up in the way described, but also that of the mass equal in magnitude to the universe."

The largest number that existed in ancient Greek arithmetic was a *myriad* or ten thousand. Archimedes introduced a new number *myriad myriad*, a hundred million, which he called *octade* or unit of the second class. *Octades octades* (or ten million billions) is called a unit of the third class, and so on.
To estimate the numbers of grains of sand in the visible Universe of that time, Archimedes had to estimate the size of the universe. This he took as a big sphere. Then he compared the size of that sphere with the size of a grain of sand and arrived at the following conclusion:

> "It is evident that the number of grains of sand that could be contained in a space as large as that bounded the the stellar sphere as estimated by Aristrachus, is not greater that one thousand myriads of units of the eight class."

(a) What is this number in modern scientific notation?

¶ **4.** As the legend goes, King Shirham of India wanted to reward his grand vizier Sissa Ben Dahir for inventing and presenting to him the game of chess. The grand vizier seemed modest in his desires: "Majesty," he said to the king, "give a grain of rice to put on the first square of this chessboard, and two grains to put on the second square, and four grains to put on the third, and eight grains to put on the fourth, and so, doubling the number for each succeeding square, give enough to cover all sisty four square on the board."

(a) How many grains of rice would have been required to fullfil the grand vizier request?

(b) How much is that in weight?

¶ **5.** The largest number that you can write with one digit is 9. The largest number that you can write with two digits (and standard math symbols) is not 99, but 9^9. What is the largest number that you can write with three digits?

¶ **6.** What is the largest number that you can write with three digits 2? You have several possibilities: 222, 2^{22}, 22^2, 2^{2^2}. Which is largest?

¶ **7.** What is the largest number that you can write with three digits 3?

¶ **8.** What is the largest number that you can write with three digits 4?

¶ **9.** The genral problem is: if a is one of the digits $1, 2, 3, 4, 5, 6, 7, 8, 9$, what is the largest number that you can write with three digits a?

¶ **10.** What is the largest number that you can write with four digits 1?

¶ 11. What is the largest number that you can write with four digits 2? There are eight possible combinations of four 2:

$$2222, \quad 222^2, \quad 22^{22}, \quad 2^{222}, \quad 22^{2^2}, \quad 2^{22^2}, \quad 2^{2^{22}}, \quad 2^{2^{2^2}}.$$

Clearly, 2222 is the smallest. To find the largest, do the following:

(a) Compare 222^2 and 22^{22}

(b) Compare 2^{222} and 22^{22}

(c) The largest of 2222, 222^2, 22^{22}, and 2^{222} is

(d) Compare this number to each of the four numbers 22^{2^2}, 2^{22^2}, $2^{2^{22}}$, $2^{2^{2^2}}$.

¶ 12. We use $m \times n$ or mn to abbreviate $\overbrace{m + m + \cdots + m}^{n}$. We also use $m \uparrow n$ to abridge m^n or $\overbrace{m \times m \times \ldots \times m}^{n}$.
Calculators print `m^n` for raising m to the power n. Old computer printouts used `m↑n` to the same effect. This suggest the following **arrow notation**, introduced by Donald Knuth in 1972:

$$
\begin{array}{rcl}
m \uparrow n & \text{abbreviates} & \underbrace{mm \cdots m}_{n} \\
m \uparrow\uparrow n & \text{abbreviates} & \underbrace{m \uparrow m \cdots \uparrow m}_{n} \\
m \uparrow\uparrow\uparrow n & \text{abbreviates} & \underbrace{m \uparrow\uparrow m \uparrow\uparrow \cdots \uparrow\uparrow m}_{n} \\
m \uparrow\uparrow\uparrow\uparrow n & \text{abbreviates} & \underbrace{m \uparrow\uparrow\uparrow m \uparrow\uparrow\uparrow \cdots \uparrow\uparrow\uparrow m}_{n}
\end{array}
$$

¶ 13. Compute the following

(a) $2 \uparrow 2$

(b) $3 \uparrow\uparrow 2$

(c) $m \uparrow\uparrow\uparrow 2$

¶ 14. The Ackermann sequence is the sequence of numbers

$$A(1) = 1 \uparrow 1, \quad A(2) = 2 \uparrow\uparrow 2, \quad A(3) = 3 \uparrow\uparrow\uparrow 3, \quad A(4) = 4 \uparrow\uparrow\uparrow\uparrow 4, \ldots$$

Compute:

(a) $A(1)$

(b) $A(2)$

(c) $A(3)$

(d) $A(4)$

Literature

[1] John Conway and Richard Guy, *The Book of Numbers*, Springer-Verlag, New York, 1996.

[2] Edward Kasner and James Newman, *Mathematics and the Imagination*, Penguin Books, 1968.

[3] George Gamov, *One, Two, Three . . . Infinity*, Dover Publications, 1988.