¶ **1.** The following problem appears to have been posed by the French lawyer and mathematician Pierre de Fermat (1601-1665) to the Italian physicist Evangelista Torricelli (1608–1647). Torricelli's student Vicenzo Viviani (1622–1703) contributed to solving this problem.

> *In the plane of a triangle find a point for which the sum of distances to the vertices of the triangle is smallest.*

This problem and others of its kind are known today as Steiner problem, after the German geometer Jakob Steiner (1796–1863).

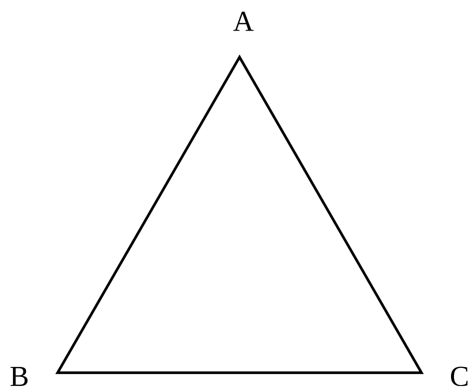
Pierre de Fermat

Evangelista Torricelli

Vicenzo Viviani

Jakob Steiner

¶ **2.** Suppose that A, B, C are the vertices of an equilateral triangle with side length s, as in the figure.



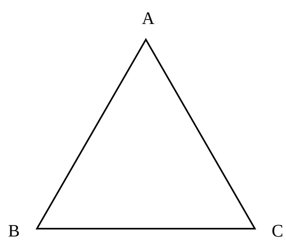
(a) What is the height of the triangle $\triangle ABC$?

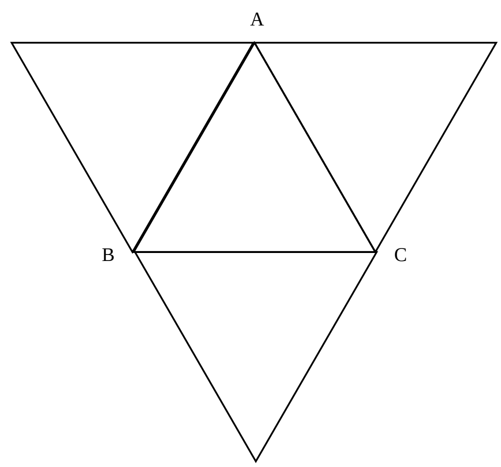
(b) Pick any point P inside the triangle and let x, y, and z be the distances from P to each of the sides AB, BC, and CA. What is $x + y + z$?

(c) What point P would you chose so the the sum of the distances from P to each of the sides of $\triangle ABC$ is smallest? (The distance from a point to a line is the length of the perpendicular segment from that point to the line.)

¶ 3. What we discover in the previous problem is known as Viviani's Theorem: *The sum of the distances from a point inside an equilateral triangle to the sides of the triangle is equal to the altitude of the triangle.* Since the distance from a point P to a line ℓ is the smallest distance from P to any point Q on the line ℓ, we can reverse Viviani's theorem and state it as follows: From a point P inside an equilateral triangle $\triangle ABC$ construct three segment to each of the three sides AB, BC and CA. Then the sum of the lengths of this segments is smallest when each segment is perpendicular to the corresponding side of the triangle.
We now analyze Steiner's problem proper for an equilateral triangle $\triangle ABC$ of side s.

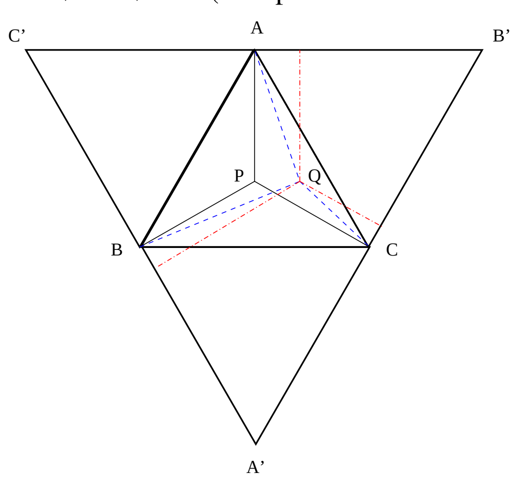
(a) Construct an equilateral triangle on each of the sides of $\triangle ABC$. Label the points A', B', C' as in the figure. What kind of triangle is $A'B'C'$?





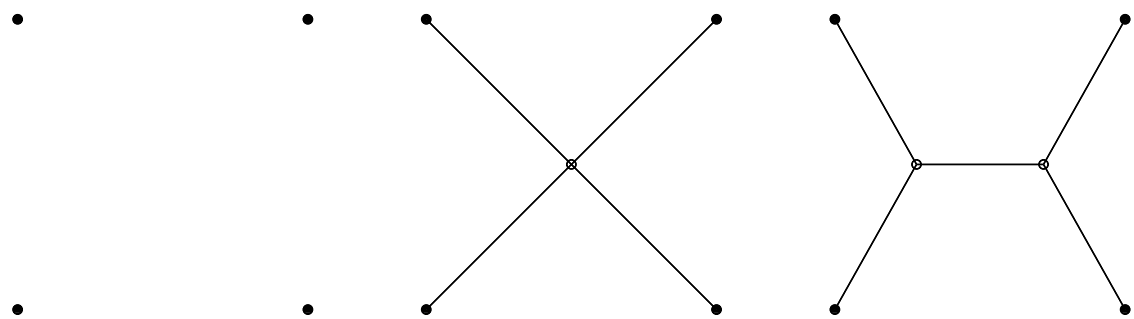
(b) Let P be the point inside the triangle ABC where the perpendicular bisectors to the sides of ABC all meet. What is the relationship between the sum of the distances $AP + BP + CP$ and the triangle $\triangle A'B'C'$?

(c) Pick any other point Q inside $\triangle ABC$. Explain why $AQ + BQ + CQ$ is at least as large than $AP + BP + CP$. (Compare this sum of distances with the sum of the lengths of the red segments.)



¶ 4. We have seen that the Steiner point of an equilateral triangle is the point where the perpendicular bisector to each of the sides of the triangle meet. Putting it this way, the solution to the equilateral triangle does not completely clarify what may be the general case. We use the Cabri Jr application to examine more cases.
The solution to finding the Steiner point of the triangle $\triangle ABC$ has in two cases: (a) if all the angles of $\triangle ABC$ are smaller than $120°$, then the Steiner point is the point inside $\triangle ABC$ subtending an angle of $120°$ with each side; (b) if $\triangle ABC$ has one vertex with angle greater than or equal to $120°$, then that vertex is the Steiner point.

¶ 5. The Steiner tree problem is to find a minimal length network that spans a set of point in the plane while allowing for the addition of auxiliary points. The case we have examined, that of three points, is the simplest non-trivial instance of the problem.
Consider a set of four points making the vertices of a square of side 1, like in the figure on the left below. What is the Steiner network for this set?



(a) Suppose that you construct a network joining the corners of the square to its center, as in the figure in the middle. What is the length of this network?

(b) Suppose that you construct a network joining the corners of the square to two other points inside the square so that the angles at those points is 120 degrees. What is the length of this network?

Literature

[1] R. Courant and H. Robbins, *What is Mathematics?* Oxford University Press, 1978

[2] Martin Gardner, *The Last Recreations*, Copernicus, 1997.