Polynomials

Formally, a polynomial is an expression of the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

The letters $a_n, a_{n-1}, \cdots, a_1, a_0$ are called the coefficients. The largest power of x is called the degree of P, the coefficient of that largest power is called the leading coefficient, and the coefficient a_0 is the constant coefficient.
For example, the polynomial $3x^4 - 4x^3 + 5x - 1$ has degree 4, coefficients $a_4 = 3$, $a_3 = -4$, $a_2 = 0$, $a_1 = 5$ and $a_0 = -1$, leading coefficient 3 and constant coefficient -1.

How to evaluate a polynomial

There are many ways in which we may write a polynomial. For example, we typically write a polynomial in descending form, like

$$P(x) = 7x^4 + x^3 - 21x^2 + 11x + 2,$$

but it can also be written in ascending form, from low to high powers of x:

$$P(x) = 2 + 11x - 21x^2 - x^3 + 7x^4,$$

or in factor form

$$P(x) = 7(x + 1/7)(x - 1)^2(x + 2)$$

because the roots of $P(x)$ are $-1/7$, 1 (a double root) and -2, and 7 is the leading coefficient.

¶ **1.** How expensive it is to evaluate a polynomial? For example, to evaluate the polynomial $2y^2 - 3y + 2$ we require a total of 12 keystrokes:

$$\boxed{2}\boxed{\times}\boxed{Y}\boxed{\times}\boxed{Y}\boxed{-}\boxed{3}\boxed{\times}\boxed{Y}\boxed{+}\boxed{2}\boxed{=}$$

How many keystrokes does it require to evaluate $P(x) = 7x^4 - x^3 - 5x^2 + 6x + 2$?

It turns out that this is not the best we can do. There is a better way which you know about. To find the value of $P(x)$ at $x = a$, we can use polynomial division to write

$$P(x) = Q(x)(x - a) + R$$

Q is the quotient of dividing P by $x - a$ (the divisor), and R is the remainder. In this case, R is just a number (because it must have degree < that the degree of $x - a$). If you plug a into x in P, then you get

$$P(a) = (a - a) + R = R$$

That is, the remainder of dividing $P(x)$ by $(x - a)$ is precisely the value of P(x) at $x = a$.

¶ 2. Is $x - 1$ a factor of $x^{567} - 3x^{400} + x^9 + 2$?

To divide a polynomial $P(x) = a_n x^n + \cdots + a_0$ by $x - c$ we use synthetic division: make a table like this:

$$\begin{array}{c|ccccccc} & a_n & a_{n-1} & a_{n-2} & \cdots & a_2 & a_1 & a_0 \\ c & & cb_{n-1} & cb_{n-2} & \cdots & cb_2 & cb_1 & cb_0 \\ \hline & q_{n-1} & q_{n-2} & q_{n-3} & \cdots & q_1 & q_0 & r \end{array}$$

Here $q_{n-1} = a_n$ and $q_k = a_{k+1} + cq_{k+1}$ are the coefficients of the quotient, and r is the remainder.

¶ 3. For example, to find the value of $P(x) = 7x^4 - 2x^3 - 9x^2 + 2x + 8$ when $x = -3$ we divide $P(x)$ by $x - (-3) = x + 3$ in this manner:

$$\begin{array}{r|rrrrr} & 7 & -2 & -9 & 2 & 8 \\ & & -21 & 69 & -180 & 534 \\ \hline -3 & 7 & -23 & 60 & -178 & 542 \end{array}$$

and obtain that $P(-3) = 542$.

¶ 4. Calculate $P(7)$ for $P(x) = 6x^7 - 40x^6 + 16x^5 - 200x^4 - 60x^3 - 69x^2 + 13x - 139$ by using synthetic division.

If you pay close attention to the calculations that you are carrying out when implementing synthetic division, you notice that what you are doing is writing $P(x) = x^4 - 2x^3 - 9x^2 + 2x + 8$ in the form

$$P(x) = (((7x - 2)x - 9)x + 2)x + 8.$$

The numbers that you see on the right side are just the coefficient of P, but now there are no powers of X directly involved.

¶ 5. To evaluate $2Y^2 - 3Y + 1$ with the calculator using this method we write $\boxed{2}\boxed{\times}\boxed{Y}\ \boxed{-}\ \boxed{3}\boxed{=}\boxed{\times}\boxed{Y}\ \boxed{=}\ \boxed{+}\ \boxed{1}\ \boxed{=}$ we also require 12 keystrokes.
How many keystrokes are required to evaluate $P(x) = 7x^4 - x^3 - 5x + 6x + 2$ using this method?

Writing a polynomial $P(x)$ in the form

$$P(x) = (\cdots(a_k x + a_{k-1})x + \cdots)x + a_0$$

makes evaluating a polynomial very amenable to programming. Note that the coefficients of P must be entered from low to high, including curly braces "`{`" and "`}`" and commas "`,`".
For example, to evaluate $P(x) = 7x^4 - x^3 - 21x^2 + 11x + 2$ at $x = 5$ with the program `pgrmEPOLY`, when prompted `COEFF P=` you input `{2,11,-21,-1,7}`

```
PROGRAM:EPOLY
:Input ''COEFF P='', ʟP
:Input ''X='',X
:0►P
:For(I,dim(ʟP),1,-1)
:P*X+ʟP(I)►P
:End
:Disp P
```

¶ 6. Can you modify the program `EPOLY` so that it evaluates a polynomial and its derivative?

Polynomial Multiplication

Multiplication of two polynomials is straightforward: we multiply each term of one by each term of the other, and the collect those resulting terms that have the same power of x. This method is very easy to implement. We imput the coefficients of P and of Q into lists ʟP and ʟQ
To create these lists, you first press [2nd] [CATALOG] [)] [ENTER] to make the little L and then you type the mane of the list. So for typing ʟP you do [2nd] [CATALOG] [)] [ENTER] [ALPHA] [8]
Then the program will create an empty list ʟPQ which will be filled by the coefficients of the product PQ by placing into the place PQ(K) the sum of all the products P(I)P(J) with I+J+1=K. The shift by 1 is because lists start at 1 instead of at 0, so the coefficient represented by the entry ʟP(J) is that of the power x^{J-1} in the polynomial P.

```
PROGRAM:MPOLY
:Input ''COEFF P='', ʟP
:Input ''COEFF Q='', ʟQ
:dim(ʟP)+dim(ʟQ)-1►dim(ʟPQ)
:For(K,1,dim(ʟPQ))
:For(I,1,dim(ʟP))
:For(J,1,dim(ʟQ))
:If K=I+J+1
:ʟPQ(K)+ʟP(I)*ʟQ(J)►ʟPQ(K)
:End
:End
:End
:Disp ʟPQ
```

Polynomial Division

If $P(x) = p_0 + p_1x + \cdots + p_mx^m$ and $D(x) = d_0 + d_1x + d_2x^2 + \cdots + d_nx^n$ are polynomials and $m \geq n$, then que can use long division to find polynomials $Q(x)$ and $R(x)$, with deg $R <$ deg D, such that

$$P(x) = Q(x)D(x) + R(x)$$

The polynomial Q is called the quotient of dividing P (the dividend) by D (the divisor), and the polynomial R is called the remainder of dividing P by D.
When $D(x) = d_0 + d_1x$ has degree 1, then we have a very easy method: synthetic division. There is also synthetic division for polynomials when the degree of the divisor is greater that 1. For example, to divide $2x^4 - 4x^3 + x^2 + 3$ by $x^3 - 2x^2 - x + 1$, we set up a table

	2	−4	1	0	3
2	*	4	0	*	*
1	*	*	2	0	*
−1	*	*	*	−2	0
	2	0	3	−2	3

The bottom left is the quotient $q(x) = 2x + 0$, and the bottom right is the remainder $3x^2 - 2 + 3$.
The method is as follows: set up a table with the coefficients of the dividend P in decreasing order on the top row, and with the coefficients of the divisor (which must have leading coefficient 1) in decreasing order on the left column, ignoring the leading coefficient, and switching their sign. Starting from the left,

(1) Add the column

(2) Multiply this column by each of the numbers on the left, starting with the uppermost, placing the products in successive columns to the right in corresponding rows.

(3) Go to (1), but omit step (2) once the quotient has been completed

¶ **7.** Use synthetic division to find the quotient and the remainder of dividing $2x^4 - 4x^3 + x^2 + 3$ by $x^2 - 3x + 4$.

The long division algorithm of polynomial that you probably know well can be easily implemented in a program for your TI-84. The algorithm for dividing $P(x)$ by $D(x)$ works as follows.

(1) Input P and D. We make sure that $\deg P = m > \deg D = n$.

(2) Multiply D by $Q = \dfrac{p_m}{d_n}x^{m-n}$.

(3) Replace P by $P - D \times Q$. This is a polynomial that has degree smaller than the degree of P in (1).

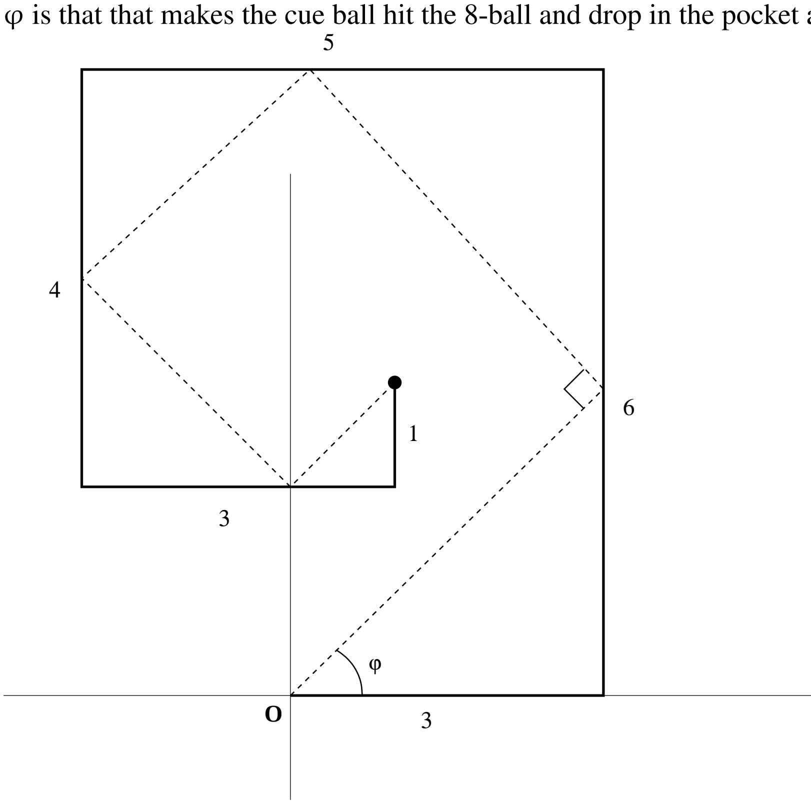
(4) If the polynomial in (3) has degree at least n, go back to (1). Otherwise, stop: this P is the remainder and the sum of all Q's in (2) is the quotient.

Here is the TI-84 program that implements this algorithm.

```
PROGRAM:DPOLY
:Input ‘‘COEFF P=’’, ʟP
:Input ‘‘COEFF D=’’, ʟD
:dim(ʟP)-dim(ʟD)+1►dim(ʟQ)
:For(K,dim(ʟQ)-1,0,-1)
:ʟP(dim(ʟD)+K)/ʟD(dim(ʟD))►ʟQ(K+1)
:For(J,dim(ʟD)+K-1,K+1,-1)
:ʟP(J)-ʟQ(K+1)*ʟD(J-K)►ʟP(J)
:End
:End
:dim(ʟD)-1►dim(ʟP)
:Disp ʟQ
:Disp ʟP
```

Visualizing the Roots: Lill's Method

Take a polynomial like $3x^5 + 6x^4 + 5x^3 + 4x^2 + 3x + 1$ and construct a polygonal path out of horizaontal and vertical segments as follows: start at the origin and move $a_5 = 3$ units to the right (because a_5 is positive). From that point, look at the sign of a_4 and walk up or down a_4 units, and repeat until the last coefficient a_0 is reached.
Now we play a funny billiard game. This polygonal path (called Lill's diagram of the polynomial) that we have drawn is the border of the billiard table. The cue ball is at the origin and the 8-ball is about to drop at the end of the path. The new thing is that in this table ball reflect on the sides with complementary angles (instead of with the same anlge as in ordinary tables), as the dotted line below indicates. The winning angle φ is that that makes the cue ball hit the 8-ball and drop in the pocket at the end of the path.



It turns out that if φ is the winning angle, then $-\tan\varphi$ is a root of $P(x)$. To verify this all you have to do is to write the polynomial $P(x)$ in the form $((((3x+6)x+5)x+4)x+3)x+1$ and use similarity of triangles.

¶ **8.** If some coefficient is 0, or some coefficient is negative, you must take that into account and be very careful when doing the polygonal paths. Find the diagrams for:

1. $x^3 + x^2 - 7x - 6$.
2. $x^3 - 7x - 6$.