Math 655. Practice Problems

May 2003

Problem 1. Find a linear transformation which carries the upper half plane onto the unit disk and the vertical axis onto the vertical diameter.

Problem 2. Find a linear transformation which carries the circle $|z| = 2$ into $|z+1| = 1$, the point -2 into the origin, and the origin into i.

Problem 3. Evaluate the following integrals

(a) $\displaystyle\int_0^\infty \frac{x \sin x}{x^2 + a^2}\, dx, \quad a$ real.

(b) $\int_0^\infty \cos x^2 \, dx$.

(c) $\displaystyle\int_0^\infty \frac{\log(x^2+1)}{x^2+1}$.

Problem 4. Suppose that f is analytic on an open connected set $U \subset \mathbf{C}$ and that $|f(z)-1| < 1$ for all $z \in U$. Prove that

$$\int_\gamma \frac{f'(z)}{f(z)}\, dz = 0$$

for every closed path γ in U.

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Problem 6. Prove that if f is an analytic mapping of the unit disk into itself which has at least two fixed points, then $f(z) = z$ for all z in the unit disk.

Problem 7. If f is an entire function and $|f(z)| = 1$ for $|z| = 1$, then $f(z) = \lambda z^n$ for some constant k with $|\lambda| = 1$ and some nonnegative integer n.

Problem 8. Show that if $|f(z)| \le 1$ for $|z| \le 1$, then

$$\frac{|f'(z)|}{1 - |f(z)|^2} \le \frac{1}{1 - |z|^2}.$$

Prove also that equality implies that f is a linear fractional transformation.

Problem 9. Let f be an analytic mapping of the unit disk $|z| < 1$ into $\Re z > 0$. Show that

$$\frac{1-|z|}{1+|z|} \le \frac{|f(z)|}{|f(0)|} \le \frac{1+|z|}{1-|z|}$$

and

$$|f'(0)| \le 2|\Re f(0)| \le 2|f(0)|.$$

(Hint: Apply Schwarz's lemma to $T \circ f$, where T is a linear transformation of the right half plane onto the unit disk sending $f(0)$ to 0.)

Problem 10. Let U be an open connected subset of $\mathbf{C}$, let $r > 0$ and let $a \in \mathbf{C}$. Show that the family $\mathcal{F}$ consisting of all functions $f \in \mathcal{A}(U)$ such that $|f(z) - a| \ge r$ for all $z \in U$ is normal in U.

Problem 11. Prove that if f is an entire function that satisfies $f(z) = f(z+1) = f(z+i)$ for all z, then f is constant.

Problem 12. Prove that every analytic mapping of the closed disk $z| \le 1$ onto itself has a fixed point.

Problem 13. Find a Riemann mapping for the following opens sets

(a) $U = \{z = x + iy \,:\, xy > 1, x > 0\}$.

(b)

(c)

Problem 14. Let $U \subset C$ be a connected open set and let $z_0 \in U$. Show that

$$J(f) = \frac{f'(z_0)}{1 - |f(z_0)|^2}$$

is a functional on the family of analytic functions f on U satisfying $|f(z)| < 1$ for all z in U.

Problem 15. A function is p-valent if it takes no value more that p times (thus, in complex analysis, one-one functions are usually called univalent functions). Show that if f_n is a sequence of p-valent analytic functions in an open an connected set U which converges uniformly on compact subsets of U to a nonconstant function f, then f is also p-valent.

Problem 16. Show that the equation $ze^{a-z} = 1$, where $a > 1$, has exactly one root in $|z| \le 1$.

Problem 17. Let f be analytic on a disk $D(z_0; r)$ with series expansion $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ for $|z - z_| < r$. Assume that $D(z_0; r)$ is contained in the upper half plane. Define $f^*(z) = \overline{f(\overline{z})}$. Show that f^* is analytic on $D(\overline{z_0}; r)$ and $f^*(z) = \sum_{n=0}^{\infty} \overline{a_n} (z - \overline{z_0})^n$ for $|z - \overline{z_0}| < r$.