Problem 3 Given the power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

with radius of convergence R, let z_0 be the zero of f which lies closest to the origin. Prove that

$$\frac{r|a_0|}{M(r) + |a_0|} \leq |z_0|$$

for any $r < R$, where $M(r) = \max_{|z|=r}\{|f(z)|\}$.

Solution Assume that $a_0 \neq 0$, otherwise $z_0 = 0$ and there is nothing to do. Then, if $|f(z) - a_0| < |a_0|$ on $|z| = \rho$, the function f has no zeros on $|z| \leq \rho$ because of Rouche's theorem, since the constant function a_0 has no zeros. Thus the problem reduces to showing that if $\rho < \dfrac{r|a_0|}{M(r) + |a_0|}$ and $|z| = \rho$, then $|f(z) - a_0| < |a_0|$.

If $r < R$, then $\dfrac{r|a_0|}{M(r) + |a_0|} < R$. Therefore, if $|z| = \rho < \dfrac{r|a_0|}{M(r) + |a_0|}$, then the following manipulations of power series are permissible.

$$\begin{aligned}
|f(z) - a_0| &\leq \sum_{n=1}^{\infty} |a_n||z|^n \\
&= \sum_{n=1}^{\infty} |a_n|\rho^n \\
&< \sum_{n=1}^{\infty} |a_n|\left(\frac{r|a_0|}{M(r) + |a_0|}\right)^n \quad \text{(series of positive coefficients)} \\
&= \sum_{n=1}^{\infty} |a_n| r^n \left(\frac{|a_0|}{M(r) + |a_0|}\right)^n \\
&\leq \sum_{n=1}^{\infty} M(r)\left(\frac{|a_0|}{M(r) + |a_0|}\right)^n \quad \text{(Cauchy's estimate)} \\
&= \left(\frac{M(r)|a_0|}{M(r) + |a_0|}\right) \sum_{n=0}^{\infty} \left(\frac{|a_0|}{M(r) + |a_0|}\right)^n \\
&= \left(\frac{M(r)|a_0|}{M(r) + |a_0|}\right) \frac{1}{1 - \dfrac{|a_0|}{M(r) + |a_0|}} \quad \text{(geometric series)} \\
&= |a_0|
\end{aligned}$$

The strict inequality appears in the third line of this string.