

## Math 655. Homework 6

**Problem 1** Let  $U \subset \mathbf{C}$  be an open set. Let  $K_n \subset U$  be the compact subset of  $U$  given by

$$K_n = \left\{ z \in \mathbf{C} : |z| \leq n, |z - w| \geq \frac{1}{n} \text{ for all } w \in \mathbf{C} \right\}$$

and for  $f, g \in \mathcal{C}(U)$  set

$$d(f, g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|f - g\|_{K_n}}{1 + \|f - g\|_{K_n}}.$$

where  $\|f\|_K = \sup\{|f(z)| : z \in K\}$ .

- (a) Prove that  $d$  is a distance function on  $\mathcal{C}(U)$ .
- (b) Prove that a sequence  $f_n \rightarrow f$  uniformly on compact subsets of  $U$  if and only if  $d(f_n, f) \rightarrow 0$  as  $n \rightarrow \infty$ .
- (c) Prove that a subset  $\mathcal{F}$  of  $\mathcal{C}(U)$  is bounded for the metric defined by  $d$  if and only if for each compact subset  $K \subset U$  there is a constant  $M$  such that  $\|f\|_K \leq M$  for all  $f \in \mathcal{F}$ .

**Definition 1** A sequence  $\{f_n\}$  of functions on  $U$  converges to  $\infty$  uniformly on compact subsets of  $U$  if for each compact set  $K \subset U$  and constant  $M > 0$  there is an integer  $N$  such that  $|f_n(z)| > M$  for all  $z$  in  $K$  and all  $n \geq N$ .

**Definition 2** A family of function  $\mathcal{F} \subset \mathcal{A}(U)$  is **normal** in  $U$  if every sequence  $\{f_n\} \subset \mathcal{F}$  contains either a subsequence which converges to a function  $f : U \rightarrow \mathbf{C}$  uniformly on compact subsets of  $U$ , or a subsequence which converges uniformly to  $\infty$  on each compact subset of  $U$ .

**Definition 3** A family of analytic functions  $\mathcal{F}$  is normal at a point  $z_0$  if there is a disk  $D(z_0; r)$  such that  $\mathcal{F}$  is normal in  $D(z_0; r)$ . (Note that this definition does not require that all the functions in the family be defined on the same set, but it does require them to be defined in the disk  $D(z_0; r)$  where normality is to hold.)

**Problem 2** Let  $\mathcal{F}$  be a family of analytic functions on the open set  $U$ . Let  $\varphi : V \rightarrow V$  be a one-one analytic function of  $V$  onto  $U$  and let  $\mathcal{G}$  be the family  $\mathcal{G} = \{f \circ \varphi : f \in \mathcal{F}\}$ . Prove that  $\mathcal{F}$  is normal in  $U$  if and only if  $\mathcal{G}$  is normal in  $V$ .

**Problem 3** Let  $\mathcal{F}$  be a family of analytic functions, and let  $U$  be an open subset of  $\mathbf{C}$ . Prove that  $\mathcal{F}$  is normal in  $U$  if and only if  $\mathcal{F}$  is normal at each point of  $U$ .

**Problem 4** (a) Let  $\mathcal{F}$  be the family  $\{f_n(z) = nz : n = 1, 2, \dots\}$ . Prove that  $\mathcal{F}$  is normal in the open set  $U \subset \mathbf{C}$  if and only if  $U$  does not contain the origin.

- (b) Let  $f : U \rightarrow \mathbf{C}$  be analytic and nowhere zero on  $U$ . Prove that the family  $\mathcal{F} = \{f_a(z) = af(z) : a > 0\}$  is normal but not compact.
- (c) Let  $\mathcal{F} = \{f_n(z) = z^n : n = 1, 2, \dots\}$ . Prove that  $\mathcal{F}$  is normal in the unit disk, but not compact.

**Problem 5** Let  $U$  be an open subset of  $\mathbf{C}$  as in the statement of the Riemann mapping theorem, let  $z_0 \in U$ , and let  $g$  be the one-one mapping of  $U$  onto the unit disk obtained in the Riemann mapping theorem.

- (a) Prove that  $g(z_0) = 0$ .
- (b) Prove that if  $f_1$  and  $f_2$  are analytic one-one mappings of  $U$  onto  $D(0; 1)$ , and  $f_1(z_0) = f_2(z_0) = 0$  and  $f_1'(z_0) = f_2'(z_0)$ , then  $f_1 = f_2$ . Deduce that  $g$  is unique subject to the constraint that  $g(z_0) = 0$  and  $g'(z_0)$  be a positive real number.
- (c) If  $f$  is an analytic mapping of  $U$  into the unit disk and  $f(z_0) = 0$ , then  $|f'(z_0)| \leq |g'(z_0)|$ , with equality if and only if  $f = \lambda g$ , for some complex number  $\lambda$  satisfying  $|\lambda| = 1$ .