Math 655. Homework 5. Due 4/9/03

The complex trigonometric functions $\sin z$ and $\cos z$ are defined by

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \qquad \text{and} \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2}.$$

Problem 1 Classify the singularities of the following functions

(a) $\dfrac{\sin^2 z}{z^4}$.

(b) $\dfrac{1}{z^2(z+1)} + \sin\dfrac{1}{z}$.

Problem 2 Find the residue at 0 of the following functions:

(a) $z\cos\dfrac{1}{z}$.

(b) $\sin\dfrac{e^z}{z}$ (express the answer in the form of an infinite series).

Problem 3 Find the number of roots of the polynomial $z^4 + 10z + 1$ in the annulus $1 < |z| < 2$.

Problem 4 Let a be a nonzero complex number and let λ be a positive real number, $\lambda \neq 1$. Find the Laurent expansion of the function $\log\dfrac{z+a}{z+\lambda a}$ at $z = 0$, where log denotes the principal branch of the logarithm which is analytic on the complement of the ray through the origin with slope λ.

Problem 5 Use the calculus of residues to evaluate the following integrals:

(a) $\displaystyle\int_0^\infty \frac{\cos x}{1+x^2}\,dx$.

(b) $\displaystyle\int_0^{2\pi} \frac{\cos\theta}{5+4\sin\theta}\,d\theta$.