Math 655. Homework 4. Solutions

Problem 1. A function $f : U \to C$ has an analytic kth root on U if there is a function h analytic on U such that $h^k = f$ on U. Suppose that f is analytic and never 0 on the open set U. Prove that f has an analytic logarithm on U if and only if f has an analytic kth root for all $k = 2, 3, \cdots$.

Solution. (1) "$\Rightarrow$" If g is an analytic logarithm for f on U, then $e^g = f$, and so, $(e^{g/n})^n = f$. Therefore $e^{g/n}$ is an analytic n-th root for f on U.

"$\Leftarrow$" We show that $\int_\gamma f'/f = 0$ for every closed path γ in U.

If γ is a closed path in U, then the path $f \circ \gamma$ does not pass throught 0 becasue f is never zero on U. Therefore,

$$\operatorname{ind}(f \circ \gamma; 0) = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, dz,$$

by a theorem in class.

Let h_n be an analytic function on U such that $(h_n)^n = f$, $n = 2, 3, \cdots$. Then

$$\frac{f'}{f} = n \frac{h_n'}{h_n}.$$

Hence

$$\frac{1}{n} \int_\gamma \frac{f'}{f} = \int_\gamma \frac{h_n'}{h_n},$$

or

$$\frac{1}{n} \operatorname{ind}(f \circ \gamma; 0) = \operatorname{ind}(h_n \circ \gamma; 0).$$

Since the index of a point with respect to a closed path is an integer, and since $\operatorname{ind}(f \circ \gamma; 0)$ does not depend on n, the only possibility for this identity to hold for all n is that $\operatorname{ind}(h_n \circ \gamma; 0) = 0$ for n sufficently large. Thus

$$\int_\gamma \frac{f'}{f} = 0$$

and so f has an analytic logarithm on U. □

Problem 2. Let a, b be two distinct complex numbers, and let U be the complement of the segment $[a, b]$. Show that $f(z) = (z - a)(z - b)$ has an analytic square root but not an analytic logarithm on U.

Problem 3. Let f, g be continuous mappings of a connected set $S \subset \mathbf{C}$ into $\mathbf{C} \setminus \{0\}$.

(1) If $f^n = g^n$ for some $n = 2, 3, \cdots$, then show that $f = e^{2\pi i k/n} g$ on S, for some $k = 0, 1, \cdots, n-1$. (Thus if f and g agree at one point, then they agree everywhere.)

(2) Show that (1) does not hold in general if f and g map into $\mathbf{C}$ instead of $\mathbf{C} \setminus \{0\}$.

Solution. (1) The function $h = f/g$ is continuous on S and $h^n(z) = 1$ for each $z \in S$. Therefore, for each $z \in S$, $h(z)$ is one of the n-th roots of 1, namely, one of the numbers $e^{2\pi ik/n}$, $k = 0, 1, \cdots, n-1$. Since h is continuous and S is connected, it follows that there is some $k = 0, 1, \cdots, n-1$ such that $h(z) = e^{2\pi ik/n}$ for all $z \in S$.
(2) Let S be the real axis and let $f(x) = x$ and $g(x) = |x|$ for $x \in S$. □

Problem 4. Give two Laurent series expansions in powers of z for the function

$$f(z) = \frac{1}{z^2(1-z)}$$

and specify the regions in which those expressions are valid.

Solution. The function f has singularities at $z = 0$ and $z = 1$. There are two Laurent series expansions, one on $0 < |z| < 1$ and another on $1 < |z| < \infty$.

First decompose into simple fractions

$$\frac{-2}{z^2(z-1)} = \frac{1}{z^2} + \frac{1}{z} + \frac{1}{1-z}$$

If $|z| < 1$, then

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$$

Therefore,

$$f(z) = \frac{-1/2}{z^2} + \frac{-1/2}{z} + \sum_{n=0}^{\infty} z^n, \qquad 0 < |z| < 1.$$

If $1 < |z|$, then $|1/z| < 1$, and

$$\frac{1}{1-z} = \frac{-1/z}{1-(1/z)} = \frac{-1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n} = \sum_{n=1}^{\infty}(-1)\frac{1}{z^n}.$$

Therefore,

$$f(z) = \frac{1}{2}z^{-2} + \frac{1}{2}z^{-1} + \sum_{-\infty}^{3}(-1)z^n, \qquad 1 < |z| < \infty.$$

□

Problem 5. Let f be analytic and never 0 on the open set $U \subset \mathbf{C}$, and let g be a continuous logarithm of f on U. Prove that g is analytic on U.

Solution. Analyticity is a local property, thus to show that f is analytic on U it suffices to show that f is analytic a disk about each point in U. Let D be a disk contained in U. Then D is convex and f is analytic and nowhere 0 on D, so it has an analytic logarithm h on D. Since g is a continuous logarithm for f on D and D is connected, $g = h + 2\pi k$, for some integer k. Thus g is analytic on D, because it is the sum of two analytic functions on D.

□