Math 655. Homework 4. Due 3/19/03

Problem 1 A function $f : U \to C$ has an analytic kth root on U if there is a function h analytic on U such that $h^k = f$ on U.

(1) Suppose that f is analytic and never 0 on the open set U. Prove that f has an analytic logarithm on U if and only if f has an analytic kth root for all $k = 2, 3, \cdots$.

(2) Let a, b be two distinct complex numbers, and let U be the complement of the segment $[a, b]$. Show that $f(z) = (z-a)(z-b)$ has an analytic square root but not an analytic logarithm on U.

Problem 2 Let f, g be continuous mappings of a connected set $S \subset \mathbf{C}$ into $C \setminus \{0\}$.

(1) If $f^n = g^n$ for some $n = 2, 3, \cdots$, then show that $f = e^{2\pi ik/n} g$ on S, for some $k = 0, 1, 2, \cdots, n-1$. (Thus if f and g agree at one point, then they agree everywhere.)

(2) Show that (1) does not hold in general if f nad g map into $\mathbf{C}$ instead of $\mathbf{C} \setminus \{0\}$.

Problem 3 Give two Laurent series expansions in powers of z for the function

$$f(z) = \frac{1}{z^2(1-z)}$$

and specify the regions in which those expressions are valid.

Problem 4 Let f be analytic and never 0 on the open set $U \subset \mathbf{C}$, and let g be a continuous logarithm of f on U. Prove that g is analytic on U.