Math 655. Homework 3. Solutions

Problem 1. Let f be an analytic function on a connected open set $U \subset \mathbf{C}$.

(1) Show that if f is real valued, then f is constant on U.

(2) Show that if f has constant absolute value, then f is constant on U.

Solution. (1) If $f = u + iv$ is real valued, then $v \equiv 0$ on U. The Cauchy-Riemann equations then imply that

$$u_x = v_y = 0 \qquad \text{and} \qquad u_y = -v_x = 0$$

on U. Therefore,

$$f' = u_x + iv_y = 0$$

and so f is constant because U is connected.

(2) If $|f|$ is constant, then $u^2 + v^2$ is constant. If this constant is 0, we are done. Otherwise, differentiating $u^2 + v^2 = \text{const.}$ and using the Cauchy-Riemann equations we obtain the system

$$\begin{cases} u_x u - u_y v &= 0 \\ u_x v + u_y u &= 0 \end{cases}$$

Since the vectors $(u(x, y), -v(x, y))$ and $(v(x, y), u(x, y))$ are linearly independent at each point $z = x + iy$ of U, the coefficients $u_x = u_y = 0$ everywhere on U. Therefore f is constant on U, as in (1). $\square$

Problem 2. Let f be analytic on $\mathbf{C}$ and real valued on $|z| = 1$. Show that f is constant.

Solution. Let $f = u + iv$. Then v is identically 0 on $|z| = 1$, hence it is identically 0 on $|z| \leq 1$, by the Maximum and Minimum principles. Because of Problem 2, f is constant on $|z| < 1$, and because of the Identity Theorem, f is constant on $\mathbf{C}$. $\square$

Problem 3. Let

$$f(z) = \int_{[1,z]} \frac{1}{w}\, dw$$

where $[1, z]$ is the line segment from 1 to z in $\mathbf{C}$. Show that f is a well defined analytic function on $\mathbf{C} \setminus \{z = x + iy \mid x \leq 0\}$, and compute its power series expansion centered at the point $z_0 = 1$.

Solution. Let $U = \mathbf{C} \setminus \{z = x + iy \mid x \leq 0\}$ and $z_0 \in U$. If h is sufficiently small in absolute value, then the triangle δ determined by the points 1, z_0 and $z_0 + h$ is contained in U. Since $1/z$ is analytic on U, Cauchy's formula for a Triangle implies that

$$\int_{\partial \triangle} \frac{1}{w}\, dw = 0$$

and so
$$f(z_0+h)-f(z_0)=\int_{[z_0+h,z_0]}\frac{1}{w}\,dw$$
.

Next
$$\begin{aligned}\left|\frac{f(z_0+h)-f(z_0)}{h}-\frac{1}{z_0}\right| &\leq \frac{1}{|h|}\int_{[z_0+h,z_0]}\left|\frac{1}{w}-\frac{1}{z_0}\right|dw\\ &\leq \max_{w\in[z_0+h,z_0]}\frac{|w-z_0|}{|wz_0|}\end{aligned}$$
which converges to 0 as $h\to 0$.

This shows that $f'(z)=1/z$. The coefficients of the power series representation $\sum_n a_n(z-1)^n$ are
$$a_n=\frac{f^{(n)}(1)}{n!}=\frac{(-1)^n}{n}.$$
□

Problem 4. Show that if $P(z)=z^n+a_{n-1}z^{n-1}+\cdots+a_0$ is a polynomial of degree $n\geq 1$, then $|P(z)|\to\infty$ as $|z|\to\infty$. In fact, show that if $|z|\geq\max\{1,2n|a_{n-1}|,\cdots,2n|a_0|\}$, then $|P(z)|\geq |z|^n/2$.

Solution. Write
$$P(z)=z^n\left(1+\frac{a_{n-1}}{z}+\cdots+\frac{a_0}{z^n}\right),$$
and let
$$M=\max\{1,2n|a_{n-1}|,\cdots,2n|a_0|\}$$
If $|z|\geq M$, then $|z|^k\geq|z|\geq M$ for all k, and
$$\frac{|a_{n-k}|}{|z^k|}\leq\frac{|a_{n-k}|}{|z|}\leq\frac{|a_{n-k}|}{2n|a_{n-k}|}=\frac{1}{2n}.$$
Therefore
$$\left|\frac{a_{n-1}}{z}+\cdots+\frac{a_0}{z^n}\right|\leq n\frac{1}{n}=\frac{1}{2}$$
and so
$$\begin{aligned}|P(z)| &\geq |z^n|\left|1+\frac{a_{n-1}}{z}+\cdots+\frac{a_0}{z^n}\right|\\ &\geq |z^n|\left(1-\left|\frac{a_{n-1}}{z}+\cdots+\frac{a_0}{z^n}\right|\right)\\ &\geq \frac{|z^n|}{2}.\end{aligned}$$
□

Problem 5. Let f be an entire function such that
$$|f(z)|\leq A|z|^k$$
for all $z\in\mathbf{C}$, for some constant A and integer k. Show that f is a polynomial of degree $\max\{0,k\}$.

Solution. If $k \leq 0$, then f is bounded and therefore constant because of Liouville's theorem.

Assume thus that $k > 0$. If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ on $\mathbf{C}$, the Cauchy inequalities and the hypothesis $|f(z)| \leq A|z|^k$ imply that

$$|a_n| \leq \frac{1}{r^n} \max_{|z|=r} |f(z)| \leq \frac{Ar^k}{r^n}$$

for all $r > 0$. Therefore $a_n = 0$ if $n > k$. $\square$