

Math 655. Homework 2. Solutions

Problem 1. Let $a \in \mathbf{C}$. Show that the function $f(z) = 1/(z - a)$ is representable by a power series on $\mathbf{C} \setminus \{a\}$.

Solution. Let $z_0 \in \mathbf{C} \setminus \{a\}$. Then

$$\begin{aligned} \frac{1}{z - a} &= \frac{-1}{(a - z_0) - (z - z_0)} \\ &= \frac{-1}{a - z_0} \frac{1}{1 - \frac{z - z_0}{a - z_0}} \\ &= \frac{-1}{a - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{a - z_0} \right)^n \\ &= \sum_{n=0}^{\infty} \frac{1}{(a - z_0)^{n+1}} (z - z_0)^n \end{aligned}$$

The ratio test implies that this series converges absolutely on $|z - z_0| < |a - z_0|$ and uniformly on compact subsets of the disk of radius $|a - z_0|$ and center z_0 . \square

Problem 2. Compute the integral

$$\int_{\gamma} \frac{1}{z^2 - 1} dz$$

where γ is the circle $|z| = 2$, oriented counterclockwise.

Solution. Write

$$\frac{2}{z^2 - 1} = \frac{1}{z - 1} - \frac{1}{z + 1}.$$

By Cauchy's formula for a circle applied to $f \equiv 1$, $z = 1$, and $|w - 1| < 2$, we obtain

$$1 = f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w - 1} = \frac{1}{2\pi i} \int_{\gamma} \frac{1}{w - 1}$$

and similarly,

$$\frac{1}{2\pi i} \int_{\gamma} \frac{1}{w + 1} = 1$$

Therefore

$$\int_{\gamma} \frac{1}{z^2 - 1} dz = 0.$$

Problem 3. Show that the following series all have radius of convergence equal to 1:

$$\sum_{n=1}^{\infty} \frac{z^n}{n^2}, \quad \sum_{n=1}^{\infty} \frac{z^n}{n}, \quad \sum_{n=0}^{\infty} z^n.$$

Show that first series converges everywhere on the unit circle; that the third series converges nowhere on the unit circle; and that the second series converges for at least one point on the unit circle and diverges for at least one point on the unit circle.

Solution. The radius of convergence is obtained by applying the ratio test. Given that $\lim n^{1/n} = 1$ (Homework 1), we have

$$\lim \left(\frac{1}{n^2} \right)^{1/n} = \left(\frac{1}{\lim n^{1/n}} \right)^2 = 1$$

and

$$\lim \left(\frac{1}{n} \right)^{1/n} = \frac{1}{\lim n^{1/n}} = 1.$$

The first series converges absolutely on $|z| = 1$. The second series diverges for $z = 1$ (it is the harmonic series $\sum 1/n$), and converges for $z = -1$ (it is the alternating series $\sum (-1/n)$).

The third series diverges everywhere on the unit circle. Indeed, if $\sum z^n$ converges for some z with $|z| = 1$, then $\lim z^n = 0$. But then $0 = |\lim z^n| = \lim |z|^n = 1$, a contradiction. \square

Problem 4. Compute the integrals

$$\int_{\gamma} \frac{e^z}{z} dz \quad \text{and} \quad \int_{\gamma} \frac{1}{1+z^2} dz$$

where γ is the counterclockwise oriented circle $|z| = 2$.

Solution. Both can be computed by applying Cauchy's Formula for a disk. For the first, let $f(z) = e^z$, which is analytic on \mathbf{C} . Cauchy's formula gives

$$f(0) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z-0} dz$$

Therefore,

$$\int_{\gamma} \frac{e^z}{z} dz = 2\pi i.$$

For the second, first write

$$\frac{1}{1+z^2} = \frac{1}{2i} \left(\frac{1}{z-i} - \frac{1}{z+i} \right)$$

Apply Cauchy's formula to each of the terms inside the parenthesis on the right side (in both cases take $f \equiv 1$) and obtain

$$\int_{\gamma} \frac{1}{z-i} = \int_{\gamma} \frac{1}{z+i} = 2\pi i.$$

Therefore

$$\int_{\gamma} \frac{1}{1+z^2} = 0.$$

\square