Math 655. Homework 2. Due 2/12/03

Problem 1 Let $a \in \mathbf{C}$. Show that the function $f(z) = 1/(z-a)$ is representable by a power series on $\mathbf{C} \setminus \{a\}$.

Can you generalize this? (If $f : U \to \mathbf{C}$ is representable by a power series on U and $f(z) \neq 0$ for every z in U, then $1/f$ is representable by a power series in U.)

Problem 2 Compute the integral

$$\int_\gamma \frac{1}{z^2-1}\,dz$$

where γ is the circle $|z| = 2$, oriented counterclockwise.

Problem 3 Let f be analytic on $D(z_0; r)$. Let $R_n(z)$ be the remainder after the term of degree n in the Taylor series expansion for f about z_0.

(a) Show that

$$R_n(z) = \frac{(z-z_0)^{n+1}}{2\pi i}\int_\Gamma \frac{f(w)}{(w-z)(w-z_0)^{n+1}}\,dw,$$

where Γ is the circle $|z - z_0| = r_1$ (oriented counterclockwise).

(b) If $|z - z_0| \leq s < r_1$, show that

$$|R_n(z)| \leq \max_{z\in\Gamma} |f(z)| \frac{r_1}{r_1 - s}\left(\frac{s}{r_1}\right)^{n+1}.$$

Problem 4 Show that the following series all have radius of convergence equal to 1:

$$\sum_{n=0}^{\infty} \frac{z^n}{n^2}, \qquad \sum_{n=0}^{\infty} \frac{z^n}{n}, \qquad \sum_{n=1}^{\infty} z^n.$$

Show that first series converges everywhere on the unit circle; that the third series converges nowhere on the unit circle; and that the second series converges for at least one point on the unit circle and diverges for at least one point on the unit circle.

Problem 5 Compute the integrals

$$\int_\gamma \frac{e^z}{z}\,dz \qquad \text{and} \qquad \int_\gamma \frac{1}{1+z^2}\,dz$$

where γ is the counterclockwise oriented circle $|z| = 2$.