Math 655. Homework 1. Solutions

Problem 1. Prove that
$$\left|\frac{a-b}{1-\overline{a}b}\right| < 1$$
if $|a| < 1$ and $|b| < 1$.

Prove also that
$$\left|\frac{a-b}{1-\overline{a}b}\right| = 1$$
if either $|a| = 1$ or $|b| = 1$. What exception must be made if $|a| = |b| = 1$?

Solution. An straightforward calculation yields
$$\left|\frac{a-b}{1-\overline{a}b}\right|^2 = \frac{|a|^2 + |b|^2 - a\overline{b} - b\overline{a}}{1 + |a|^2|b|^2 - a\overline{b} - b\overline{a}}$$
Thus we are required to show that
$$|a|^2 + |b|^2 - a\overline{b} - b\overline{a} < 1 + |a|^2|b|^2 - a\overline{b} - b\overline{a}$$
or
$$|a|^2 + |b|^2 < 1 + |a|^2|b|^2,$$
if $|a| < 1$, $|b| < 1$.

If $|a| < 1$ and $|b| < 1$, then $|a|^2 < 1$ and $1 - |b|^2 > 0$, so
$$|a|^2(1 - |b|^2) < (1 - |b|^2),$$
which is equivalent to the previously displayed inequality.

If $|a| = 1$ but $|b| < 1$, then
$$\left|\frac{a-b}{1-\overline{a}b}\right|^2 = \frac{1 + |b|^2 - a\overline{b} - b\overline{a}}{1 + |b|^2 - a\overline{b} - b\overline{a}} = 1$$

In case $|a| = |b| = 1$, the exception to be made is $1 - \overline{a}b = 0$, or $a = b$ (because $\overline{a} = 1/a$ and $\overline{b} = 1/b$ when $|a| = |b| = 1$). □

Problem 2. Show that the functions $f(z)$ and $\overline{f(\overline{z})}$ are simultaneously analytic.

Solution. A function f is differentiable at z if and only if the limit

$$\lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$$

exists. Now, if $g(z) = \overline{f(\overline{z})}$, then

$$\frac{g(z+h)-g(z)}{h} = \frac{\overline{f(\overline{z}+\overline{h})}-\overline{f(\overline{z})}}{h} = \overline{\left(\frac{f(\overline{z}+\overline{h})-f(\overline{z})}{\overline{h}}\right)}$$

Since $h \to 0$ if and only if $\overline{h} \to 0$ and conjugation commutes with limits, it follows that g is differentiable at z if and only if f is differentiable at $\overline{z}$. In fact, $g'(z) = \overline{f'(\overline{z})}$.

The problem can also be solved using the Cauchy-Riemann equations. The Jacobian of a function $f = f(x, y) = u(x, y) + iv(x, y)$ is the 2×2 matrix Df given by

$$Df = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

For instance, the Jacobian of the function $z \mapsto iz$ (multiplication by i) is the matrix

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and the Jacobian of the conjugation map $r(z) = \overline{z}$ is

$$R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The Cauchy-Riemann equations for $f = u + iv$ can be written thus

$$Df \cdot J = J \cdot Df.$$

The function $g(z) = \overline{f(\overline{z})}$ is the composition $g = r \circ f \circ r$, and so its Jacobian is

$$Dg = R \cdot Df \cdot R$$

Since $R \cdot J = -J \cdot R$, we have that, if f satisfies the Cauchy-Riemann equations, then

$$J \cdot Dg = J \cdot R \cdot Df \cdot R = -R \cdot J \cdot Df \cdot R = -R \cdot Df \cdot J \cdot R = R \cdot Df \cdot R \cdot J = Dg \cdot J$$

so g also satisfies the Cauchy-Riemann equations, and conversely. □

Problem 3. Evaluate the integral

$$\int_\gamma \overline{z}\, dz,$$

where γ is the arc of the parabola $y = x^2$ from $(1, 1)$ to $(2, 4)$.

Solution. Parametrize the curve by $\gamma(t) = t + it^2$, $1 \le t \le 2$. Then

$$\begin{aligned}
\int_\gamma \overline{z}\, dz &= \int_1^2 \overline{\gamma(t)}\gamma'(t)\, dt \\
&= \int_1^2 (t - it^2)(1 + 2it)\, dt \\
&= \left[\frac{t^2}{2} + i\frac{t^3}{3} + \frac{t^4}{2}\right]_1^2 \\
&= 9 + \frac{7}{3}i
\end{aligned}$$

□

Problem 4. If the power series $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence R, show that the differentiated series $\sum_{n=1}^{\infty} n a_n z^{n-1}$ also has radius of convergence R.

Solution. The radius of convergence R of the series $\sum_{n=0}^{\infty} a_n z^n$ is given by

$$\frac{1}{R} = \limsup_n |a_n|^{1/n}.$$

To show that the radius of convergence of $\sum_{n=1}^{\infty} n a_n z^{n-1}$ is also R, it thus suffices to show that $\lim_n n^{1/n} = 1$. This is done in the book. For $n > 1$, $n^{1/n} = 1 + x_n$ for some $x_n > 0$. Then $n = (1 + x_n)^n = 1 + nx_n + \cdots \ge 1 + nx_n$. Therefore $0 \le x_n \le (n-1)/n \to 0$ as $n \to \infty$. □

Problem 5. A function f defined on an open set U has a primitive on U if there is a function $F : U \to \mathbf{C}$ such that $F' = f$ on U. Show that the function $f(z) = 1/z$ has no primitive on $0 < |z| < 1$.

Solution. We have proved in class that if f has a primitive on U, then $\int_\gamma f = 0$ for every closed path γ in U.

Let $\gamma(t) = e^{it}/2$, $0 \le t \le 2\pi$. Then

$$\begin{aligned}
\int_\gamma \frac{1}{z}\, dz &= \int_0^{2\pi} \frac{1}{e^{it}/2} ie^{it}/2\, dt \\
&= \int_0^{2\pi} i\, dt \\
&= 2\pi i
\end{aligned}$$

□