

Chapter 3

The Cauchy Formula

3.1 Logarithms

The exponential function was described in Section 1.6. This is an analytic mapping $\exp : \mathbf{C} \rightarrow \mathbf{C}$ whose image is the plane minus the origin. A branch of the logarithm on an open subset U of $\mathbf{C} \setminus \{0\}$ is a map $h : U \rightarrow \mathbf{C}$ such that $\exp h(z) = z$. The elementary branches of the logarithm are of the form $\log z = \ln |z| + i\theta(z)$, where $\theta(z)$ is the argument of z in $(\theta_0, \theta_0 + 2\pi]$.

In general, there are analytic functions which are branches of the logarithm but do not coincide with one of the elementary branches. For example, let U be the open set depicted in the figure and define

$$h(z) = \begin{cases} \ln |z| + i\theta(z), & 0 \leq \theta < 2\pi, \text{ for } z \in U_1 \\ \ln |z| + i\theta(z), & \pi \leq \theta < 3\pi, \text{ for } z \in U_2 \end{cases}$$

Locally h coincides with one of the elementary branches of $\log z$, so it is an analytic branch of \log on U . However, a single branch will not suffice since any elementary branch will be discontinuous on a ray through the origin, and any such ray intersects U .

Definition 3.1.1. Let $S \subset \mathbf{C}$ and $f : S \rightarrow \mathbf{C} \setminus \{0\}$ be continuous. The function $g : S \rightarrow \mathbf{C}$ is a *continuous logarithm* of f if g is continuous on S and $\exp g(z) = f(z)$ for all $z \in S$. The function $\theta : S \rightarrow \mathbf{R}$ is a *continuous argument* of f if θ is continuous on S and $f(z) = |f(z)| \exp(i\theta(z))$ for all $z \in S$.

Continuous logarithms are closely related to continuous arguments.

Theorem 3.1.2. Let $S \subset \mathbf{C}$ and $f : S \rightarrow \mathbf{C} \setminus \{0\}$ be continuous.

- (a) If g is a continuous logarithm of f , then $\Im g$ is a continuous argument of f .
- (b) If θ is a continuous argument of f , then $\ln |f(z)| + i\theta(z)$ is a continuous logarithm.
- (c) If S is connected and g_1, g_2 are continuous logarithms of f , then $g_1 - g_2 = 2\pi in$ for some integer n ; if θ_1, θ_2 are continuous arguments of f , then $\theta_1 - \theta_2 = 2\pi m$ for some integer m .

(d) If S is connected and $z, w \in S$, then

$$g(w) - g(z) = \ln |f(w)| - \ln |f(z)| + i\theta(w) - i\theta(z)$$

for all continuous logarithms g and continuous arguments θ of f .

Proof. (a) $f = \exp g = \exp(\Re g + i\Im g) = |f| \exp \Im g$.

(b) $\exp(\ln |f| + i\theta) = |f|e^{i\theta} = f$.

It follows that a non vanishing continuous function on a set S has a continuous logarithm if and only if it has a continuous argument.

(c) If $e^{g_1} = e^{g_2} = f$, then $g_1 - g_2$ is an integer multiple of $2\pi i$. Since S is connected, this multiple is the same in all of S . Similarly if $e^{i\theta_1} = e^{i\theta_2} = f/|f|$.

(d) By (b), $\ln |f| + i\theta$ is a continuous logarithm of f . By (c), $g = \ln |f| + i\theta + 2\pi in$ for some integer n , and the result follows. \square

In general, a given function may not have a continuous logarithm.

Exercise 3.1.3. Let $S = \{|z| = 1\}$ and $f(z) = z$. Show that f does not have a continuous argument function on S .

Solution. Let θ be a continuous argument of f on S , so that $z = e^{i\theta(z)}$ for $|z| = 1$. Let $\gamma(t) = e^{it}$, $0 \leq t \leq 2\pi$. Then $e^{it} = \exp i\theta(e^{it})$. Then t and $\theta(e^{it})$ are both continuous arguments of γ , hence $\theta(e^{it}) = t + 2\pi k$ for some integer k . Let $t \rightarrow 0$ to obtain $\theta(1) = 2\pi k$ and let $t \rightarrow 1$ to obtain $\theta(1) = 1 + 2\pi k$, and contradiction. \square

However, if S is a closed interval $[a, b]$, then f will have a continuous logarithm.

Theorem 3.1.4. Let $f : [a, b] \rightarrow \mathbf{C} \setminus \{0\}$ be continuous. Then f has a continuous logarithm on S .

Proof. Suppose that the image of f is contained in a disk D which does not contain 0. Then there is an analytic branch of the logarithm function $\log z$ defined on D , that is, there is a function h analytic on D such that $\exp h(z) = z$. Therefore, $g(t) = h(f(t))$ is an analytic logarithm of f .

In general, cover the image of f by disks like above, and patch up logarithms. This is done as follows. Since $[a, b]$ is compact, $|f|$ has a minimum $m > 0$ on $[a, b]$. Form a partition $a = t_0 < \dots < t_n = b$ such that $|f(t) - f(t_j)| < m$ on $[t_j, t_{j+1}]$ for all j . Then $f(t) \in D = D(f(t_j), m)$ for t in $[t_j, t_{j+1}]$, and $0 \notin D$ by definition of m . With h as above we have $e^{h(f(t))} = f(t)$ on t_j, t_{j+1} . If $e^{g_0(t)} = f(t)$ for $t_0 \leq t \leq t_1$ and $e^{g_2(t)} = f(t)$ for $t_1 \leq t \leq t_2$, then $g_0(t_1) = g_1(t_1) + 2\pi in$. Replace g_1 by $g_1 + 2\pi in$ on $[t_1, t_2]$. This produces a continuous logarithm of f on $[t_0, t_1]$. Repeat the process inductively to obtain a continuous logarithm of f on $[a, b]$. \square

Definition 3.1.5. Let f be analytic and never 0 on the open set U . A function $g : U \rightarrow \mathbf{C}$ is an *analytic logarithm* of f on U if g is analytic on U and $\exp g(z) = f(z)$ for all z in U .

Theorem 3.1.6. Let f be analytic and never 0 on U . Then f has an analytic logarithm on U if and only if the function f'/f has a primitive on U .

Proof. If $\exp g = f$ on U , then the chain rule implies $g' \exp g = f'$, and hence $f'/f = g'$.

Conversely, if g is analytic on U and $g' = f'/f$, then $f \exp(-g)$ has zero derivative on U and thus $f \exp -g$ is equal to a constant k_A on each component A of U . Choose c_A so that $e^{c_A} = k_A$ (the constant $k_A \neq 0$). Then $\exp(g + c_A) = f$ on A . \square

Analytic logarithms exist under fairly general conditions.

Theorem 3.1.7. *Let $U \subset \mathbf{C}$ be a convex open set, and let f be analytic and never 0 on U . Then f has an analytic logarithm on U . More generally, if U is an open subset such that $\int_{\gamma} h = 0$ for every closed path in U and every analytic function h , then every analytic function f which is never 0 on U has an analytic logarithm on U .*

Proof. First consider the convex case. If f'/f is analytic on U , then Cauchy's Theorem for a Convex Set (Theorem 2.2.6) yields $\int_{\gamma} f'/f = 0$ for every closed path γ in U . By Theorem 3.1.6, f has an analytic logarithm. \square

Theorem 3.1.8. *If g is an analytic logarithm of f on the open set U , and γ is a path in U , then*

$$\int_{\gamma} \frac{f'(z)}{f(z)} dz = g(\gamma(b)) - g(\gamma(a))$$

where $[a, b]$ is the parameter interval of γ .

Proof. By the argument of Theorem 3.1.6, $g' = f'/f$ on U . The result follows at once. \square

3.2 The Index of a point with respect to a curve

Definition 3.2.1. If X is a topological space, a *curve in X* is a continuous mapping γ of a compact interval $[a, b]$ in \mathbf{R} into X , here $a < b$. We call $[a, b]$ the *parameter interval* of γ and denote the range of γ by γ^* . Thus γ is a mapping, and γ^* is the set of all points of the form $\gamma(t)$, for $a \leq t \leq b$. We say that γ is a closed curve if $\gamma(a) = \gamma(b)$.

Theorem 3.2.2. *Let γ be a closed curve such that $0 \notin \gamma^*$. Let θ be a continuous argument of γ . Then*

$$\frac{1}{2\pi}(\theta(b) - \theta(a))$$

is an integer, independent of the continuous argument chosen.

Proof. For each $t \in [a, b]$, $e^{i\theta(t)} = \gamma(t)/|\gamma(t)|$. Thus

$$e^{i\theta(b) - i\theta(a)} = \frac{\gamma(b)}{|\gamma(b)|} \frac{|\gamma(a)|}{\gamma(a)} = 1,$$

since γ is closed. It follows that $\theta(b) - \theta(a)/2\pi$ is an integer.

If ϕ is another continuous argument, then $\phi(t) = \theta(t) + 2\pi m$ for some integer m , and the result follows. \square

Definition 3.2.3. Let γ be a closed curve, and z_0 a point not in γ^* . Let θ be a continuous argument of the curve $\gamma - z_0$. The index of z_0 with respect to γ is the integer

$$\text{ind}(z_0; \gamma) = \frac{\theta(b) - \theta(a)}{2\pi}.$$

Intuitively, $\text{ind}(\gamma; z_0)$ is the net number of revolutions of $\gamma(t)$ about z_0 , so sometimes it is called the winding number.

Lemma 3.2.4. *Let γ be a curve, $V \subset \mathbf{C}$ an open subset which contains γ^* . Then there is a partition $a = t_0 < \cdots < t_n = b$ and disks $D_1, \dots, D_n \subset V$ such that $\gamma[t_{j-1}, t_j] \subset D_j$, for all $j = 1, \dots, n$.*

Proof. Let $\varepsilon = \text{dist}(\gamma^*, \mathbf{C} \setminus V) > 0$. By uniform continuity of γ , there is a $\delta > 0$ such that $|t - t'| < \delta$ implies $|\gamma(t) - \gamma(t')| < \varepsilon$. Let $a = t_0 < t_1 < \cdots < t_n = b$ be a partition of $[a, b]$ such that $|t_j - t_{j-1}| < \delta$ for all j . Let $D_j = D(\gamma(t_j); \varepsilon) \subset V$, $j = 1, \dots, n$. If $t_{j-1} \leq t \leq t_j$, then $|t - t_j| < \delta$ and $|\gamma(t) - \gamma(t_j)| < \varepsilon$, and thus $\gamma(t) \in D_j$. \square

Theorem 3.2.5 (Index of a Point with Respect to a Path). *Let γ be a closed path, z_0 a point not in γ^* . Then*

$$\text{ind}(\gamma; z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{1}{z - z_0} dz$$

More generally, if f is analytic on γ^ and $z_0 \notin (f \circ \gamma)^*$, then*

$$\text{ind}(f \circ \gamma; z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z) - z_0} dz$$

Proof. Because γ^* is compact, there is an open set V containing γ^* such that $f - z_0$ is never 0 on V . (Indeed, because the set $(f \circ \gamma)^*$ is compact and does not contain z_0 , there is an open set U which contains $(f \circ \gamma)^*$ but not z_0 . It suffices to take an open set $V \subset f^{-1}(U)$ containing γ^* where f is defined.) Construct a partition $a = t_0 < \cdots < t_n = b$ and disks $D_1, \dots, D_n \subset V$ as in Lemma 3.2.4. Then $f - z_0$ has an analytic logarithm g_j on each D_j , and so, if γ_j is the restriction of γ to $[t_{j-1}, t_j]$, then

$$\int_{\gamma_j} \frac{f'(z)}{f(z) - z_0} dz = g_j(\gamma(t_j)) - g_j(\gamma(t_{j-1})).$$

Let $\theta(t)$ be a continuous argument $f(\gamma(t)) - z_0$ on $[a, b]$. Because $g_j(\gamma(t))$ is a continuous logarithm of $f(\gamma(t)) - z_0$ on $[t_{j-1}, t_j]$, it follows that

$$\begin{aligned} \frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z) - z_0} dz &= \frac{1}{2\pi i} \sum_{j=1}^n g_j(\gamma(t_j)) - g_j(\gamma(t_{j-1})) \\ &= \frac{1}{2\pi i} \sum_{j=1}^n \left[\ln |f(\gamma(t_j)) - z_0| - \ln |f(\gamma(t_{j-1})) - z_0| \right. \\ &\quad \left. + i\theta(t_j) - i\theta(t_{j-1}) \right] \\ &= \frac{\theta(b) - \theta(a)}{2\pi} \\ &= \text{ind}(f \circ \gamma; z_0) \end{aligned}$$

\square

Finally we consider the behavior of $\text{ind}(\gamma; z)$ as z varies.

Theorem 3.2.6. *If γ is a closed path, then $\text{ind}(\gamma; z)$, regarded as a function of z , is constant on each component of $\mathbf{C} \setminus \gamma^*$, and equal to 0 on the unbounded component.*

Proof. The function $\text{ind}(\gamma; \bullet)$ is analytic, hence continuous on $C \setminus \gamma^*$. But on any given component of $\mathbf{C} \setminus \gamma^*$, $\text{ind}(\gamma; \bullet)$ is a continuous, integer valued function, and is therefore constant.

Since γ^* is compact, it lies on a bounded disk D whose complement D^c is connected; thus D^c lies in some component of $\mathbf{C} \setminus \gamma^*$. This shows that $\mathbf{C} \setminus \gamma^*$ has precisely one unbounded component.

By Theorem 2.2.13, $\text{ind}(\gamma; z) \rightarrow 0$ as $z \rightarrow \infty$, hence $\text{ind}(\gamma; \bullet)$ must be 0 on the unbounded component of $\mathbf{C} \setminus \gamma^*$. \square

3.3 Cauchy's Theorem

The statement of the general form of Cauchy's Theorem requires to integrate over objects slightly more general than closed paths.

Definition 3.3.1. A *cycle* is a formal sum $\sigma = n_1\gamma_1 + \cdots + n_k\gamma_k$, where the a_i are integers and the γ_i are closed paths. The union $\bigcup_{j=1}^k \gamma_j^*$ of the images of the individual paths forming σ is denoted by σ^* .

If f is a continuous function on γ^* , then define

$$\int_{\gamma} f = \sum_{j=1}^k a_j \int_{\gamma_j} f.$$

A cycle γ is said to be *equivalent to 0* if

$$\int_{\gamma} f = 0$$

for all continuous functions f on γ^* . Two cycles γ_1 and γ_2 are *equivalent* if the difference $\gamma_1 - \gamma_2$ is equivalent to 0.

Finally, the index of a point z with respect to a cycle $\gamma = \sum_{j=1}^n n_j\gamma_j$ is defined to be

$$\text{ind}(\gamma; z) = \sum_{j=1}^k n_j \text{ind}(\gamma_j; z).$$

If $U \subset \mathbf{C}$ is an open set, and γ is a cycle with $\gamma^* \subset U$, then γ is called a cycle in U . Cycles can be added in the obvious way, thus the space of cycles forms a commutative group, denoted by $Z(U)$.

Definition 3.3.2. A cycle σ on U is homologous to 0 on U , written $\sigma \sim 0 \pmod{U}$ if $\text{ind}(\sigma; z) = 0$ for all z not in U .

It is clear that the space of cycles in U which are homologous to 0 modulo U is a subgroup of the group of cycles in U . This subgroup is denoted by $B(U)$, and the quotient group $Z(U)/B(U) = H(U)$ is called the first homology group of U with integer coefficients.

Theorem 3.3.3. *Let U be a connected open subset of \mathbf{P} . Then $H_1(U)$ is a free commutative group of rank equal to the number of components of the complement of U in \mathbf{P} minus one.*

The following technical lemma will be very useful for subsequent developments.

Lemma 3.3.4 (Fundamental Lemma). *Let γ be a closed polygonal path whose edges are parallel to the coordinate axis. For a mesh consisting of all lines parallel to the axes which passes through the vertices of γ . The complement of this mesh consists of regions of three types: bounded rectangular regions R_1, \dots, R_k , unbounded regions having three sides, and unbounded regions having two sides. Chose points $z_j \in R_j$, and let γ_0 be the cycle*

$$\gamma_0 = \sum_{j=1}^k \text{ind}(\gamma; z_j) \partial R_j$$

where ∂R_j is the closed path determined by the boundary of R_j , oriented counterclockwise. Then γ and γ_0 are equivalent.

Proof. First note that

$$(1) \quad \text{ind}(\gamma_0; z_k) = \sum_{j=1}^m \text{ind}(\gamma; z_j) \text{ind}(\partial R_j; z_k) = \text{ind}(\gamma; z_k),$$

for $k = 1, \dots, m$. Also, if $z'_k \in R'_k$,

$$\begin{aligned} \text{ind}(\gamma_0; z'_k) &= \sum_{j=1}^m \text{ind}(\gamma; z_j) \text{ind}(\partial R_j; z'_k) \\ &= 0 \\ &= \text{ind}(\gamma; z'_k) \end{aligned}$$

because z'_k belongs to the unbounded component of $\mathbf{C} \setminus \gamma^*$.

Now suppose that σ_{ij} is an edge lying between R_i and R_j . Suppose that in $\gamma - \gamma_0$, the edge σ_{ij} is traversed c times (c is an integer, possibly negative). Let $\sigma = \gamma - \gamma_0 - c\partial R_i$. Then

$$(3) \quad \text{ind}(\sigma; z_i) = \text{ind}(\gamma; z_i) - \text{ind}(\gamma_0; z_i) - c \text{ind}(\partial R_i, z_i) = -c$$

by (1). Also by (1),

$$(4) \quad \text{ind}(\sigma; z_j) = \text{ind}(\gamma; z_j) - \text{ind}(\gamma_0; z_j) - c \text{ind}(\partial R_i, z_j) = 0.$$

But σ_{ij} essentially does not appear in σ , that is, σ is equivalent to a cycle τ in which σ_{ij} does not appear; thus $\text{ind}(\sigma; z_k) = \text{ind}(\tau; z_k)$ for all k by Theorem 3.2.5 and the definition of equivalent cycles.

Since z_i and z_j belong to the same component of $\mathbf{C} \setminus \tau^*$, the indices $\text{ind}(\tau; z_i) = \text{ind}(\tau; z_j)$. By (3) and (4), $c = 0$ so that σ_{ij} contributes nothing to $\gamma - \gamma_0$. Exactly the same argument, with z_j replaced by z'_j , shows that if σ'_{ij} is an edge lying between R_i and an unbounded region R'_j , then σ'_{ij} contributes nothing to $\gamma - \gamma_0$. But all edges of $\gamma - \gamma_0$ are of the form σ_{ij} or σ'_{ij} , hence $\gamma - \gamma_0$ is equivalent to 0. \square

Theorem 3.3.5. *If f is analytic on the open set $U \subset \mathbf{C}$, then $\int_{\gamma} f = 0$ for every cycle γ in U such that $\text{ind}(\gamma; z) = 0$ for every z in the complement of U .*

Proof. Construct a partition $a = t_0 < t_1 < \cdots < t_n = b$ of the parameter domain of γ and disks $D_1, \dots, D_k \subset U$ such that $\gamma(t) \in D_j$ for $t_{j-1} \leq t \leq t_j$. If γ_j is a polygonal path in D_j from $z_{j-1} = \gamma(t_{j-1})$ to $z_j = \gamma(t_j)$ with edges parallel to the axes, then the integral of f from z_{j-1} to z_j along γ is the same as the integral along γ_j . Thus we may assume without loss of generality that γ is a polygonal cycle with sides parallel to the axes. By the lemma, γ is equivalent to a cycle of the form $\sum n_k \text{ind}(\gamma; z_j) \partial R_j$, where R_j is a rectangle with sides parallel to the axes, and $z_j \in R_j$. If R_j is one of these rectangles and $\overline{R_j}$ is not contained in U , let $z_0 \in \overline{R_j} \setminus U$. Since z_0 is not in γ^* , the segment $[z_0, z_j]$ does not intersect γ^* , hence z_0 and z_j lie in the same component of $\mathbf{C} \setminus \gamma^*$, and so $\text{ind}(\gamma; z_j) = \text{ind}(\gamma; z_0)$. By hypothesis, $\text{ind}(\gamma; z_0) = 0$. Therefore, γ is equivalent to a cycle of the form $\sigma = \sum_{j=1}^l \text{ind}(\gamma; z_j) \partial R_j$, where all $\overline{R_j} \subset U$.

By the equivalence of γ and σ and by Cauchy's Theorem for a Convex Set (Theorem 2.2.6) we obtain that

$$\int_{\gamma} f = \sum_{j=1}^l \text{ind}(\gamma; z_j) \int_{\partial R_j} f = 0,$$

as advertised. □

Theorem 3.3.6 (First Cauchy Theorem). *Let $U \subset \mathbf{C}$ be an open subset and let σ be a cycle in U . Then $\int_{\sigma} f = 0$ for every analytic function on U if and only if $\text{ind}(\sigma; z) = 0$ for every $z \notin U$.*

Proof. The “if” part is Theorem 3.3.5. If γ is a cycle in U and $\int_{\gamma} f = 0$ for every analytic function on U , then $\text{ind}(\gamma; z) = 0$ for every z in the complement of U . For if $z_0 \notin U$ and $\text{ind}(\gamma; z_0) \neq 0$, let $f(z) = 1/(z - z_0)$. Then f is analytic on U and, by Theorem 3.2.5, $\int_{\gamma} f = 2\pi i \text{ind}(\gamma; z_0) \neq 0$. □

Proposition 3.3.7. *Let U be an open subset of \mathbf{C} . The following properties are equivalent.*

- (1) *If σ is a cycle in U , then $\text{ind}(\sigma; z) = 0$ for every point z in the complement of U .*
- (2) *The integral $\int_{\sigma} f = 0$ for every cycle σ in U and every function f analytic on U .*
- (3) *If f is analytic on U , then f has a primitive on U .*
- (4) *If f is analytic on U and f has no zeros on U , then f has an analytic logarithm on U .*

Proof. (1) implies (2). This is First Cauchy Theorem.

(2) implies (3). This is the Existence of Primitive Theorem (Theorem 2.2.4).

(3) implies (4). If f is analytic and has no zeros, then f'/f is analytic on U , and hence has a primitive on U . Theorem 3.1.6 then implies that f has an analytic logarithm on U .

(4) implies (1). If $z_0 \notin U$, then $f(z) = z - z_0$ is analytic and never 0 on U , hence it has an analytic logarithm on U . Now

$$\begin{aligned} \text{ind}(\sigma; z_0) &= \frac{1}{2\pi i} \int_{\sigma} \frac{1}{z - z_0} dz \\ &= 0, \end{aligned}$$

by Theorem 3.1.6, since $f'/f = 1/(z - z_0)$. □

These properties of an open set U can be described by yet another property which is somewhat more topological.

Definition 3.3.8. An open set $U \subset \mathbf{C}$ is simply connected if both U and \mathbf{P} are connected.

Theorem 3.3.9. *A connected open set U is simply connected if and only if the index $\text{ind}(\sigma; z) = 0$ for every cycle σ in U and every point z in the complement of U .*

Proof. Suppose that $\mathbf{P} \setminus U = A \cup B$, the union of two disjoint open sets. One of these sets, say B , contains the point at infinity, and thus the other is bounded. Let $\text{dist}(A, B) = \delta > 0$. Cover the plane by a net of squares of side $< \delta/\sqrt{2}$. Choose the net so that certain point $a \in A$ lies at the center of one of the squares. Consider the cycle

$$\sigma = \sum_k \partial R_k$$

where the sum is taken over all of the squares of the net which have a point in common with A . It is clear that σ is disjoint from B , by reasons of distance. It is clear that once cancellations are carried out, the cycle σ is equivalent to a cycle which does not intersect A . Finally, since a belongs to exactly one of the squares making up σ , the index $\text{ind}(\sigma; a) = 1$.

Conversely, assume that U is simply connected. Let γ be a closed path in U . Defining $\text{ind}(\gamma; \infty) = 0$, then $\text{ind}(\gamma; \bullet)$ is a continuous function on $\mathbf{P} \setminus \gamma^*$. Since $\gamma^* \subset U$, $\mathbf{P} \setminus U$ is a connected subset of $\mathbf{P} \setminus \gamma^*$; also $\infty \in \mathbf{P} \setminus U$ since $U \subset \mathbf{C}$. Therefore, if $z \notin U$, then z and ∞ belong to the same component of $\mathbf{P} \setminus \gamma^*$, therefore, $\text{ind}(\gamma; z) = \text{ind}(\gamma; \infty) = 0$. \square

The general form of Cauchy's theorem can now be given.

Theorem 3.3.10 (Cauchy's Integral Formula). *Let f be analytic on the open set U . Let σ be a cycle in U such that $\text{ind}(\sigma; z) = 0$ for every z in the complement of U . If $z \in U \setminus \gamma^*$, then*

$$f(z) \text{ind}(\sigma; z) = \frac{1}{2\pi i} \int_{\sigma} \frac{f(w)}{w - z} dw.$$

Proof. Let $g : U \rightarrow \mathbf{C}$ be defined by

$$g(w) = \begin{cases} \frac{f(w) - f(z)}{w - z} & \text{if } w \neq z, \\ f'(z) & \text{otherwise.} \end{cases}$$

Theorem 2.2.11 implies that g is analytic on U and Theorem 3.3.5 implies that $\int_{\sigma} g = 0$. Therefore,

$$\begin{aligned} \frac{1}{2\pi i} \int_{\sigma} \frac{f(w)}{z - w} dz &= f(z) \frac{1}{2\pi i} \int_{\sigma} \frac{1}{w - z} dw \\ &= f(z) \text{ind}(\sigma; z) \end{aligned}$$

by Theorem 3.2.5. \square