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Chapter 1

Preliminaries

1.1 Basic Definitions

The *complex plane* \mathbf{C} is the set of all ordered pairs (a, b) of real numbers, with addition and multiplication defined by

$$(a, b) + (c, d) = (a + c, b + d)$$

$$(a, b)(c, d) = (ac - bd, ad + bc).$$

If $i = (0, 1)$ and the real number a is identified with $(a, 0)$, then $(a, b) = a + bi$. The expression $a + bi$ can be manipulated as if it were an ordinary binomial expression of real numbers, subject to the condition $i^2 = -1$. With the above definitions of addition and multiplication, \mathbf{C} is a field.

If $z = a + bi$, then a is called the *real part* of z , written $a = \Re z$, and b is called the *imaginary part* of z , written $b = \Im z$. The *absolute value*, or modulus, or magnitude, of z is defined as $(a^2 + b^2)^{1/2}$. An *argument* of $z = a + bi$ (written $\arg z$) is defined as the angle which the line segment from $(0, 0)$ to (a, b) makes with the positive real axis. The argument is not unique, but it is determined up to a multiple of 2π . (The argument of 0 may be defined arbitrarily.) The *principal argument* of z , denoted by $\text{Arg } z$, is the argument in the interval $(-\pi, \pi]$.

If r is the modulus of z and θ is an argument of z , we can write

$$z = r \cos \theta + ir \sin \theta$$

and it follows from trigonometric identities that

$$|zw| = |z||w|$$

and

$$\arg(zw) = \arg z + \arg w$$

(that is, if θ_z is an argument of z and θ_w is an argument of w , then $\theta_z + \theta_w$ is an argument of zw). If $w \neq 0$, then $\arg(z/w) = \arg z - \arg w$.

If $z = a + bi$, the *conjugate* of z is defined as $\bar{z} = a - bi$, and the following properties hold:

- (1) $|\bar{z}| = |z|$
- (2) $\arg \bar{z} = -\arg z$
- (3) $\overline{z + w} = \bar{z} + \bar{w}$

$$(4) \overline{z\bar{w}} = \overline{z}\bar{w}$$

$$(5) 2\Re z = z + \bar{z}$$

$$(6) 2i\Im z = z - \bar{z}$$

$$(7) z\bar{z} = |z|^2$$

The *distance* between two complex numbers z and w is defined as

$$d(z, w) = |z - w|$$

Thus d is simply the usual Euclidean distance between z and w regarded as points in the plane. Thus d defines a *metric* on \mathbf{C} , and furthermore \mathbf{C} is complete, that is, every Cauchy sequence converges. If z_n is a sequence of complex numbers, then $z_n \rightarrow z$ if and only if $\Re z_n \rightarrow \Re z$ and $\Im z_n \rightarrow \Im z$.

If z_0 is a complex number and $r \geq 0$, then the set of complex numbers z such that $|z - z_0| < r$ is called the (open) disk of radius r and center z_0 , and it denoted by $D(z_0; r)$.

1.2 The Extended Plane

Consider the complex plane \mathbf{C} embedded in \mathbf{R}^3 as the plane spanned by the first two axes. In coordinates, a complex number $z = x_1 + ix_2$ is identified with the point $(x_1, x_2, 0)$ in \mathbf{R}^3 .

The set of unit vectors in \mathbf{R}^3 is the unit 2-sphere S^2 ,

$$S^2 = \{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 + x_3^2 = 1\}.$$

The stereographic projection, h_N , of the 2-sphere onto the complex plane \mathbf{C} is defined as follows. Let $N = (0, 0, 1)$ be the north pole of S^2 . If p is another point in the 2-sphere, define $h_N(p)$ to be the point in \mathbf{C} obtained as the intersection of the line through N and p with \mathbf{C} . This defines a mapping

$$h_N : S^2 \setminus \{N\} \rightarrow \mathbf{C}$$

called the stereographic projection.

Consider the set of pairs of complex numbers (z, w) having the property that not both are 0. Declare two pairs (z, w) and (z', w') to be equivalent if there is a non-zero complex number λ such that $\lambda z = z'$ and $\lambda w = w'$. The set of equivalence classes is denoted by \mathbf{P} , and is precisely the set of lines in the vector space \mathbf{C}^2 . A general element of \mathbf{P} , that is, all pairs equivalent to a given pair (z, w) will be denoted by $[z, w]$.

There are two obvious maps of \mathbf{C} into \mathbf{P} , namely, $\phi_0(z) = [z, 1]$ and $\phi_1(z) = [1, z]$. These maps are one to one, and their common domain is the set of non-zero complex numbers

1.3 Analytic functions

Let $f : U \rightarrow \mathbf{C}$, where U is an open subset of \mathbf{C} . Then f is said to be analytic on U if and only if for every $z \in U$, $(f(z+h) - f(z))/h$ approaches a limit $f'(z) \in \mathbf{C}$ as $h \rightarrow 0$ in \mathbf{C} . This means that given $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that if $|h| < \delta$, then

$$\left| \frac{f(z+h) - f(z)}{h} - f'(z) \right| < \epsilon.$$

Analyticity on a non-open set $S \subset \mathbf{C}$ means analyticity on an open set $U \supset S$. In particular, f is analytic at a point z_0 if and only if f is analytic on an open set U with $z_0 \in U$.

If f and g are analytic on U , so are $f + g$, $f - g$, kf for $k \in \mathbf{C}$, fg , and f/g (provided g is never 0 on U). Furthermore, the usual rules for differentiation of functions of a real variable apply, with basically the same proof. This means that $\mathcal{H}(U)$, the space of analytic functions on U , is a ring.

Exercise 1.3.1. If f is analytic on U , then f is continuous on U . This is proved as in the real variable case.

By direct calculation, the identity function $z \mapsto z$ is analytic on \mathbf{C} and $z' = 1$. Using the product rule,

$$\frac{d}{dz}(z^n) = nz^{n-1} \quad n = 0, 1, 2, \dots$$

using the quotient rule this extends to $n = -1, -2, \dots$. More generally, the function $f(z) = 1/z^n$ is analytic on $z \neq 0$, as it follows by direct calculation.

Example 1.3.2. Let $f(z) = 1/(z - a)$. Then, if $z_0 \neq a$ and $|z - z_0| < |z_0 - a|$, then,

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{1}{(z - a)(z_0 - a)}$$

which converges to $1/(z_0 - a)^2$ as $z \rightarrow z_0$.

Example 1.3.3. Let $f(z) = \bar{z}$. Then

$$\frac{f(z+h) - f(z)}{h} = \frac{\bar{h}}{h}$$

which has no limit as $h \rightarrow 0$.

If f is analytic on U and g is analytic on $f(U)$ (that is, if g is analytic on an open set $V \subset \mathbf{C}$ such that $f(U) \subset V$), then the composition $g \circ f$ is analytic on U and

$$(g \circ f)'(z) = g'(f(z))f'(z),$$

just as in the real case. Indeed, call $h = g \circ f$, fix $z_0 \in U$ and put $w_0 = f(z_0)$. Then

$$f(z) - f(z_0) = [f'(z_0) + \alpha(z)](z - z_0)$$

$$g(w) - g(w_0) = [g'(w_0) + \beta(w)](w - w_0),$$

where $\alpha(z) \rightarrow 0$ as $z \rightarrow z_0$ and $\beta(w) \rightarrow \beta(w_0)$ as $w \rightarrow w_0$. Put $w = f(z)$ and substitute the second identity into the third. Then, if $z \neq z_0$,

$$\frac{h(z) - h(z_0)}{z - z_0} = [g'(f(z_0)) + \beta(f(z))] [f'(z_0) + \alpha(z)].$$

The differentiability of f forces f to be continuous at z_0 ; therefore

$$\lim_{z \rightarrow z_0} \frac{h(z) - h(z_0)}{z - z_0} = g'(f(z_0))f'(z_0),$$

as desired.

1.4 Cauchy-Riemann Equations

Let $f : U \rightarrow \mathbf{C}$, where U is an open subset of \mathbf{C} . Let u be the real part of f , and v the imaginary part of f , so that

$$f(z) = f(x + iy) = u(x, y) + iv(x, y)$$

If f is analytic on U , the *Cauchy-Riemann equations* hold on U

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Also

$$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}$$

Conversely, if $f = u + iv$ is defined on U and u and v have continuous first partial derivatives and satisfy the Cauchy-Riemann equations on U , then f is analytic on U .

A similar statement can be made about differentiability at a point. If f is defined in a neighborhood of z_0 and is differentiable at z_0 , then the Cauchy-Riemann equations hold at z_0 . Conversely, if the first partial derivatives of u and v exist in a neighborhood of z_0 , are continuous at z_0 , and the Cauchy-Riemann equations hold at z_0 , then f is differentiable at z_0 .

Example 1.4.1. (1) Let $f(z) = |z|$. Then $u(x, y) = (x^2 + y^2)^{1/2}$ and $v(x, y) = 0$. The function f is continuous everywhere and differentiable nowhere.

(2) Let $f(z) = |z|^2$. This is continuous everywhere, differentiable at 0 but nowhere else.

(3) Let $f(z) = |xy|^{1/2}$. All the partial derivatives of u and v are 0 at $z = 0$, so the Cauchy-Riemann equations hold at that point. However, f is not differentiable at $z = 0$ (approach along $x = y$). The partial derivatives are discontinuous at the origin.

If $f = u + iv$ is analytic on the open set U , then u and v satisfy the Laplace equation

1.5 Power series

The only result from power series that we need at this point is that given a power series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ there exists a number $0 \leq R \leq \infty$ such that the series converges absolutely on $|z - z_0| < R$ and diverges on $|z - z_0| > R$. Furthermore, the series converges uniformly on compact subsets of $|z - z_0| < R$. Therefore, $f(z) = \sum a_n(z - z_0)^n$ is a continuous function on $|z - z_0| < R$.

This number R is called the *radius of convergence*, and is given by the root test:

$$\frac{1}{R} = \limsup_{n \rightarrow \infty} |a_n|^{1/n}$$

(with the standard convention that $1/0 = \infty$ and $1/\infty = 0$).

Theorem 1.5.1. Suppose that the power series $\sum_{n=0}^{\infty} a_n z^n$ converges for some $z_0 \neq 0$. Then if $|z| < |z_0|$, the two series $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=1}^{\infty} n a_n z^n$ converge absolutely.

Proof. If the series $\sum a_n z_0^n$ converges, then the sequence $\{a_n z_0^n\}$ converges to 0 and is therefore bounded. Thus there exists $M > 0$ such that $|a_n z_0^n| \leq M$ for all n . Then

$$\begin{aligned} |a_n z^n| &= |a_n z_0^n| \left| \frac{z}{z_0} \right|^n \\ &\leq M \left| \frac{z}{z_0} \right|^n \end{aligned}$$

Since the series $\sum_{n=0}^{\infty} |z/z_0|^n$ converges, the series $\sum a_n z^n$ converges absolutely. \square

Theorem 1.5.2. For any power series

$$\sum_{n=0}^{\infty} a_n z^n$$

one of the following three possibilities must be true:

- (1) The series converges only for $z = 0$.
- (2) The series converges absolutely for all z in \mathbf{C} .
- (3) There exists a number $R > 0$ such that the series converges absolutely if $|z| < R$ and diverges if $|z| > R$.

Proof. Let $S = \{x \in \mathbf{R} \mid \sum_{n=0}^{\infty} a_n w^n \text{ converges for some } w \text{ with } |w| = x\}$. Suppose that S is unbounded. Then for every complex number z there exists $x \in S$ such that $|z| < x$, which, by the definition of S , means that $\sum_{n=0}^{\infty} a_n w^n$ converges for some w with $|w| = x > |z|$. By Theorem ??, the series $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely.

Suppose now that S is bounded, and let R be the least upper bound of S . If $R = 0$, then $\sum_{n=0}^{\infty} a_n z^n$ converges only for $z = 0$. If $R > 0$, then given a complex number z with $|z| < R$, there is a number x in S with $|z| < x$. As before, this means that the series $\sum_{n=0}^{\infty} a_n w^n$ converges for some w with $|w| = x > |z|$, so that $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely. Moreover, if $|z| > R$, then $\sum_{n=0}^{\infty} a_n z^n$ does not converge since $|z|$ is not in S . \square

Definition 1.5.3. The power series $\sum_{n=0}^{\infty} a_n w^n$ converges uniformly on the set S if there is a function $B : S \rightarrow \mathbf{C}$ such that for each $\varepsilon > 0$ there is an integer N such that

$$\left| \sum_{k=0}^n a_k z^k - B(z) \right| \leq \varepsilon$$

for all $z \in S$ and all $n \geq N$.

Theorem 1.5.4. If the series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges at the point z , where $|z - z_0| = r$, then the series converges absolutely on $|z - z_0| < r$, and uniformly on each closed subdisk of $D(z_0; r)$, hence uniformly on compact subsets of $D(z_0; r)$.

Proof. Apply the Weierstrass M-test. \square

Definition 1.5.5. A function f defined on U is representable by a power series in U if to every disk $D(z_0; r) \subset U$ there corresponds a series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ which converges to $f(z)$ for all z in $|z - z_0| < r$.

Theorem 1.5.6. *If f is representable by a power series in U , then f is analytic in U . In fact, if*

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$$

for z in $D(z_0; r)$, then for these z we also have

$$f'(z) = \sum_{n=1}^{\infty} n a_n(z - z_0)^{n-1}$$

Proof. The power series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ converges in $|z - z_0| < r$, and the root test shows that the series $\sum_{n=1}^{\infty} n a_n(z - z_0)^{n-1}$ also converges there. Take $z_0 = 0$, without loss of generality, and let $g(z)$ denote the sum of the series $\sum_{n=1}^{\infty} n a_n(z - z_0)^{n-1}$. Fix w in $D(0; r)$, and choose ρ so that $|w| < \rho < r$. If $z \neq w$, then

$$\frac{f(z) - f(w)}{z - w} - g(w) = \sum_{n=1}^{\infty} a_n \left[\frac{z^n - w^n}{z - w} - n w^{n-1} \right].$$

The expression in brackets is 0 if $n = 1$, and is

$$(z - w) \sum_{k=1}^{n-1} k w^{k-1} z^{n-k-1}$$

if $n \geq 2$. If $|z| < \rho$, the absolute value of the sum above is less than

$$\frac{n(n-1)}{2} \rho^{n-2}$$

so

$$\left| \frac{f(z) - f(w)}{z - w} - g(w) \right| \leq |z - w| \sum_{n=2}^{\infty} n^2 |a_n| \rho^{n-2}.$$

Since $\rho < r$, the last series converges. Hence the left side of this inequality tends to 0 as $z \rightarrow w$. This says that $f'(w) = g(w)$, and completes the proof. \square

Since f' satisfies the same hypothesis as f does, the theorem can be applied to f' . Therefore,

Corollary 1.5.7. *If f is representable by a power series in U , then f has derivatives of all orders, and each is also representable by a power series in U . In fact*

$$f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1) \cdots (n-k+1) a_n (z - z_0)^{n-k}$$

on $|z - z_0| < r$ if $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ on $|z - z_0| < r$.

In particular, for such f , the coefficients of the series are given by

$$a_k = \frac{f^{(k)}(z_0)}{k!} \quad k = 0, 1, 2, \dots$$

so that the series is uniquely determined by f .

1.6 The Elementary Functions

The *exponential function* is defined as follows. If $z = x + iy$, define

$$e^z = e^x(\cos y + i \sin y).$$

Then

$$e^{z+w} = e^z e^w$$

The exponential function can be defined by the power series

$$e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$$

The Cauchy-Riemann equations show that $z \mapsto e^z$ is analytic on all of \mathbf{C} and $d/dz(e^z) = e^z$. Alternatively, note that

$$e^z = 1 + z \left(\sum_{n=1}^{\infty} \frac{1}{(n+1)!} z^n \right) = 1 + zf(z)$$

and that the series representing $f(z)$ in parenthesis converges in all of \mathbf{C} , with $f(0) = 1$. Therefore,

$$\frac{e^{z+h} - e^z}{h} = e^z \frac{e^h - 1}{h} = e^z f(h)$$

which converges to e^z as $h \rightarrow 1$.

Theorem 1.6.1. *The exponential function satisfies the following properties.*

- (a) $\exp z \neq 0$ for all z .
- (b) The derivative $\exp' = \exp$.
- (c) $\exp(z + 2\pi i) = \exp z$
- (d) $t \mapsto e^{it}$ maps the reals onto the unit circle.
- (e) If $w \neq 0$ is a complex number, then there exists z such that $e^z = w$.

Proof. If $w \neq 0$, let $\alpha = w/|w|$. Since $|w| > 0$, there is a real number x such that $e^x = |w|$. Since $|\alpha| = 1$, there is a real number y such that $e^{iy} = \alpha$. Therefore, $e^{x+iy} = |w|\alpha = w$. \square

If k is an integer, the exponential function maps the strip $\{(2k-1)\pi < y \leq (2k+1)\pi\}$ (in fact, any horizontal strip of width 2π) one-to-one onto $\mathbf{C} \setminus \{0\}$, and thus there are infinitely many ways of defining the inverse of the exponential.

The sine and cosine functions are defined by

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad \text{and} \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

The usual trigonometric identities and differentiation formulas hold.

1.7 Logarithms and Roots

If $z \neq 0$, define a logarithm of z by $\log z = \ln |z| + i(\theta + 2k\pi)$, where k is an integer and θ , the argument of z is chosen in $(\theta_0, \theta_0 + 2\pi]$, where θ_0 is a fixed real number.

Then $e^{\log z} = z$, so for each k , the corresponding branch of the logarithm is an inverse of the exponential. $\log z$ maps $\mathbf{C} \setminus 0$ one-to-one onto the strip

$$\{u + iv \mid \theta_0 + 2k\pi < v \leq \theta_0 + (2k+2)\pi\}.$$

The logarithm is analytic on $\mathbf{C} \setminus L$, where L is a ray from the origin with angle θ_0 . All the branches have the same derivative, $1/z$.

If $k = 0$ and $\theta_0 = -\pi$, the resulting branch is called the principal branch, and denoted by $\text{Log } z$.

Exercise 1.7.1. Compute $\text{Log } i$.

Solution. The absolute value $|i| = 1$ and the principal argument $\text{Arg } i = \pi/2$. Thus $\text{Log } i = i\pi/2$. \square

Exercise 1.7.2. If D is a disk in \mathbf{C} which does not contain the origin, then show that there is an analytic branch of the logarithm defined on D , that is, show that there is an analytic function h on D such that $\exp h(z) = z$ for all $z \in D$.

For a complex number α define

$$z^\alpha = e^{\alpha \log z} = |z|^\alpha e^{i\alpha(\theta + 2k\pi)}$$

If $\alpha = m/n$ is a rational number (in lowest terms), there are only n branches (all analytic on $\mathbf{C} \setminus L$). In particular,

$$z^{1/n} = |z|^{1/n} e^{i(\theta + 2k\pi)/n}$$

for $k = 0, 1, \dots, n-1$.

For each k , the corresponding branch of the n th root maps \mathbf{C} one-to-one onto the sector

$$\left\{ r e^{i\theta} \mid r \geq 0, \frac{\theta_0 + 2k\pi}{n} < \theta \leq \frac{\theta_0 + 2(k+1)\pi}{n} \right\}$$

Define $0^{1/n} = 0$.

For the principal branch ($k = 0, \theta_0 = -\pi$) the sector is defined by $-\pi/n < \theta \leq \pi/n$. A particular branch of $z^{1/n}$ has derivative $(1/n)z^{(1/n)-1} = (1/n)z^{1/n}/z$ on $\mathbf{C} \setminus L$.

Chapter 2

The Basic Theory

2.1 Integration on Paths

Definition 2.1.1. A *path* is a continuous, piecewise continuously differentiable map γ from a bounded closed interval $[a, b]$ of the reals to \mathbf{C} .

If $\gamma(a) = \gamma(b)$, then γ is called a closed path. If γ is a path, its image $\gamma[a, b]$ will be denoted by γ^* . If $\gamma^* \subset U$, then γ is said to be a path in U .

Piecewise continuously differentiable for a path γ with parameter interval $[a, b]$ means that there are finitely many points t_j , $a = t_0 < t_1 < \cdots < t_n = b$ such that the restriction of γ to each interval $[t_j, t_{j+1}]$ has continuous derivative on $[t_j, t_{j+1}]$; however, at the points t_1, \dots, t_{n-1} , the left- and right-handed derivatives of γ may differ.

Definition 2.1.2. The *length* of the path γ defined over the interval $[a, b]$ is

$$\text{length}(\gamma) = \int_a^b |\gamma'(t)| dt.$$

Definition 2.1.3. Let γ be a path with domain $[a, b]$ and let $f : \gamma^* \rightarrow \mathbf{C}$ be a continuous function. The *integral of f on γ* is defined by

$$\int_{\gamma} f(z) dz = \int_a^b f(\gamma(t)) \gamma'(t) dt.$$

Let ϕ be a piecewise continuously differentiable one-to-one mapping of an interval $[a_1, b_1]$ onto $[a, b]$ so that $\phi(a_1) = a$ and $\phi(b_1) = b$, and put $\gamma_1 = \gamma \circ \phi$. Then γ_1 is a path with parameter interval $[a_1, b_1]$, called a reparametrization of γ ; the integral of f over γ_1 is

$$\begin{aligned} \int_{a_1}^{b_1} f(\gamma_1(t)) \gamma_1'(t) dt &= \int_{a_1}^{b_1} f(\gamma(\phi(t))) \gamma'(\phi(t)) \phi'(t) dt \\ &= \int_a^b f(\gamma(t)) \gamma'(t) dt. \end{aligned}$$

so that the “reparametrization” of γ has not changed the integral:

$$\int_{\gamma_1} f(z) dz = \int_{\gamma} f(z) dz.$$

Whenever this identity holds for a pair of paths γ_1 and γ (and all f), we shall regard γ_1 and γ as equivalent.

It is convenient to be able to replace a path by an equivalent one, i.e., to choose parameter intervals at will. For instance, if the end point of γ_1 coincides with the initial point of γ_2 , we may relocate their parameter intervals so that γ_1 and γ_2 join to form a path γ , with the property that

$$\int_{\gamma} f = \int_{\gamma_1} f + \int_{\gamma_2} f$$

for every continuous f on $\gamma^* = \gamma_1^* \cup \gamma_2^*$.

However, suppose that $[0, 1]$ is the parameter interval of a path γ , and $\gamma_1(t) = \gamma(1-t)$, $0 \leq t \leq 1$. Then γ_1 is called *the path opposite to γ* , for the following reason: if f is continuous on $\gamma_1^* = \gamma^*$, then

$$\begin{aligned} \int_0^1 f(\gamma_1(t))\gamma_1'(t) dt &= - \int_0^1 f(\gamma(1-t))\gamma'(1-t) dt \\ &= - \int_0^1 f(\gamma(s))\gamma'(s) ds, \end{aligned}$$

so that

$$\int_{\gamma_1} f = - \int_{\gamma} f$$

Lemma 2.1.4 (Basic Integral Estimate). *Let γ be a path in \mathbf{C} and f be continuous on γ^* . Then*

$$\left| \int_{\gamma} f(z) dz \right| \leq \sup_{z \in \gamma^*} |f(z)| \text{length}(\gamma).$$

Example 2.1.5. (1) If a is a complex number and $r > 0$, then the path $\gamma(t) = a + re^{it}$, $0 \leq t \leq 2\pi$, is called the *positively oriented circle* with center a and radius r . We have

$$\int_{\gamma} f(z) dz = ir \int_0^{2\pi} f(a + re^{i\theta}) e^{i\theta} d\theta$$

and the length of γ is $2\pi r$.

(2) If z_1 and z_2 are points of \mathbf{C} , then the path defined by $\gamma(t) = (1-t)z_1 + tz_2$, $0 \leq t \leq 1$, is the oriented interval $[z_1, z_2]$; its length is $|z_2 - z_1|$, and

$$\int_{[z_1, z_2]} f(z) dz = (z_2 - z_1) \int_0^1 f(z_1 + (z_2 - z_1)t) dt.$$

If

$$\gamma_1(t) = \frac{(b-t)z_1 + (t-a)z_2}{b-a} \quad (a \leq t \leq b)$$

then γ_1 is an equivalent path, which will be still denoted by $[z_1, z_2]$. The opposite path to $[z_1, z_2]$ is $[z_2, z_1]$.

(3) Let $\{z_1, z_2, z_3\}$ be an ordered triple of complex numbers, let $\Delta = \Delta(z_1, z_2, z_3)$ be the triangle with vertices z_1 , z_2 and z_3 , and define

$$\int_{\partial\Delta} f = \int_{[z_1, z_2]} f + \int_{[z_2, z_3]} f + \int_{[z_3, z_1]} f$$

for any f continuous on the boundary of Δ .

(4) More generally, if Γ is an oriented polygon with vertices labeled $z_0, z_1, \dots, z_n = z_0$, then

$$\int_{\Gamma} f(z) dz = \sum_{k=0}^{n-1} \int_{[z_k, z_{k+1}]} f(z) dz.$$

The following theorem computes an elementary but useful integral.

Theorem 2.1.6. For $n = 0, 1, \dots$, the integral

$$\int_{[z_1, z_2]} z^n dz = \frac{z_2^{n+1} - z_1^{n+1}}{n+1}$$

Proof. Let $\gamma(t) = (1-t)z_1 + tz_2$, $0 \leq t \leq 1$; then

$$\begin{aligned} \int_{[z_1, z_2]} z^n dz &= \int_0^1 ((1-t)z_1 + tz_2)^n (z_2 - z_1) dt \\ &= \left[\frac{((1-t)z_1 + tz_2)^{n+1}}{n+1} \right]_0^1 \\ &= \frac{z_2^{n+1} - z_1^{n+1}}{n+1} \end{aligned}$$

as claimed. □

Theorem 2.1.7. Let $f(z) = z^n$ and let $\gamma(t) = e^{it}$, $0 \leq t \leq 2\pi$. Then

$$\int_{\gamma} f(z) dz = \begin{cases} 0 & \text{if } n \neq -1 \\ 2\pi & \text{if } n = -1 \end{cases}$$

Proof. The derivative $\gamma'(t) = ie^{it}$; hence, if $n \neq -1$,

$$\begin{aligned} \int_{\gamma} f(z) dz &= \int_0^{2\pi} e^{i(n+1)t} i dt \\ &= \frac{1}{n+1} e^{i(n+1)t} \Big|_0^{2\pi} \\ &= 0. \end{aligned}$$

□

2.2 The Local Cauchy Theorem

There are several forms of Cauchy's theorem. They all assert that if γ is a closed path in U , and γ and U satisfy certain topological conditions, then the integral of every analytic function f over γ is 0. We will first derive a simple local version of this theorem which is quite useful for many applications.

Theorem 2.2.1. Suppose that f is analytic in U and that f' is continuous in U . Then

$$\int_{\gamma} f'(z) dz = 0$$

for every closed path in U .

Proof. If $[a, b]$ is the parameter interval of γ , the fundamental theorem of calculus shows that

$$\begin{aligned}\int_{\gamma} f'(z) dz &= \int_a^b f'(\gamma(t))\gamma'(t) dt \\ &= f(\gamma(b)) - f(\gamma(a)) \\ &= 0\end{aligned}$$

since $\gamma(a) = \gamma(b)$. □

Theorem 2.2.2 (Cauchy's Theorem for a Triangle). *Suppose that Δ is a closed triangle in the open set U . If f is analytic on U , then*

$$\int_{\partial\Delta} f(z) dz = 0$$

Proof. Subdivide the triangle Δ into 4 triangles Δ_{0j} with boundary $\partial\Delta_{0j}$, the vertices of Δ_{04} being the midpoints of the vertices of Δ . Then

$$\int_{\partial\Delta} f(z) dz = \sum_{j=1}^4 \int_{\partial\Delta_{0j}} f(z) dz$$

Choose k such that $\int_{\partial\Delta_{0k}} f$ has the largest magnitude, and call $\Delta_1 = \Delta_{0k}$, etcetera. Then

$$\left| \int_{\partial\Delta} f \right| \leq 4 \left| \int_{\partial\Delta_1} f \right|$$

Continuing this process, we obtain a nested sequence of triangles $\Delta \supset \Delta_1 \supset \dots$ such that

$$\left| \int_{\partial\Delta} f \right| \leq 4^n \left| \int_{\partial\Delta_n} f \right|$$

Furthermore, since the length of the segment joining the midpoints of two sides of a triangle has length equal to half that of the opposite side, the diameter of $\Delta_n \leq \text{length } \partial\Delta_n = 2^{-n} \text{length } \partial\Delta$. Therefore the diameter of $\Delta_n \rightarrow 0$ as $n \rightarrow \infty$, and thus the intersection $\bigcap_n \Delta_n$ consists of exactly one point $z_0 \in U$.

Let $g(z) = f(z) - f(z_0) - (z - z_0)f'(z_0)$. Since f is analytic at z_0 , given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|g(z)| < \varepsilon |z - z_0|,$$

for $0 \leq |z - z_0| < \varepsilon$. If n is sufficiently large, then $\Delta_n \subset \{|z - z_0| < \delta\}$, and so

$$\int_{\partial\Delta_n} f(z) dz = \int_{\partial\Delta_n} [f(z_0) - (z - z_0)f'(z_0)] dz + \int_{\partial\Delta_n} g(z) dz$$

The first integral on the right is zero by Theorem 2.1.6, and the absolute value of the second integral is, by Lemma 2.1.4,

$$\begin{aligned}\left| \int_{\partial\Delta_n} g \right| &\leq \varepsilon \max\{|z - z_0| \mid z \in \partial\Delta_n\} (\text{length } \partial\Delta_n) \\ &\leq \varepsilon (\text{length } \partial\Delta_n)^2 \\ &= \varepsilon 4^{-n} (\text{length } \partial\Delta)^2\end{aligned}$$

Therefore,

$$\left| \int_{\partial\Delta} f(z) dz \right| \leq \varepsilon (\text{length } \partial\Delta)^2,$$

and since ε is arbitrary the theorem is proved. \square

Definition 2.2.3. Let $f : U \rightarrow \mathbf{C}$. The function $F : U \rightarrow \mathbf{C}$ is said to be a *primitive* of f on U if $F' = f$ on U .

Theorem 2.2.4. Let f be a continuous function on the open set U . Then f has a primitive on U if and only if $\int_{\gamma} f = 0$ for every closed path in U .

Proof. If $F' = f$ on U and $[a, b]$ is the parameter interval of γ , the fundamental theorem of calculus shows that

$$\begin{aligned} \int_{\gamma} F'(z) dz &= \int_a^b F'(\gamma(t)) \gamma'(t) dt \\ &= F(\gamma(b)) - F(\gamma(a)) \\ &= 0 \end{aligned}$$

since $\gamma(a) = \gamma(b)$.

Conversely, assume that $\int_{\gamma} f = 0$ for every closed path γ in U . First let U be connected and fix $z_0 \in U$. Define

$$F(z) = \int_{z_0}^z f(w) dw$$

where the integral is taken along any path in U from z_0 to z . The hypothesis of the theorem guarantees that such integral is independent of the path. For h sufficiently small in absolute value,

$$\left| \frac{F(z+h) - F(z)}{h} - f(z) \right| = \left| \frac{1}{h} \int_{[z, z+h]} [f(w) - f(z)] dw \right|$$

This is bounded above by

$$\max\{|f(w) - f(z)| \mid w \in [z+h, z]\}$$

which converges to 0 as $h \rightarrow 0$ because f is continuous. If U is not connected, then apply the above argument to each component of U . \square

If U is convex, then triangles are adequate in Theorem 2.2.4.

Theorem 2.2.5. Let f be a continuous map $f : U \rightarrow \mathbf{C}$, where U is a convex open subset of \mathbf{C} . If $\int_{\partial\Delta} f = 0$ for every triangle Δ in U , then f has a primitive in U , so that $\int_{\gamma} f = 0$ for every closed path γ in U .

Proof. Fix $z_0 \in U$ and set

$$F(z) = \int_{[z_0, z]} f(w) dw$$

Note that $[z_0, z] \subset U$ because U is convex. The argument of Theorem 2.2.4 shows that $F' = f$ on U . \square

Theorem 2.2.6 (Cauchy's Theorem for a Convex Set). *Suppose that Δ is a closed triangle in the open subset U of \mathbf{C} , $z_0 \in U$, f is continuous on U and analytic on $U \setminus \{z_0\}$. Then*

$$\int_{\partial\Delta} f(z) dz = 0$$

In particular, on account of Theorem 2.2.5, if U is convex, then $\int_\gamma f = 0$ for every closed path in U .

Proof. Let V be the interior of Δ . If $z_0 \notin \Delta$, then $\int_{\partial\Delta} f = 0$, by Theorem 2.2.2. If z_0 is a vertex of Δ , then there is a sequence of points $z_n \in \Delta$ such that $z_n \rightarrow z_0$. Let Δ_n be the triangle determined by z_n and the side of Δ not containing z_0 . Then, because f is uniformly continuous on Δ ,

$$\int_{\partial\Delta_n} f \rightarrow \int_{\partial\Delta} f$$

as $n \rightarrow \infty$.

If $z_0 \in \Delta$ but it is not a vertex, subdivide Δ into three triangles, each having z_0 as a vertex, and apply the reasoning of the previous paragraph. \square

Remark. It will be shown later in Theorem 2.2.11 that if f is continuous on U and analytic on $U \setminus \{z_0\}$, then f is necessarily analytic on U . However, the weaker formulation of the hypothesis will simplify the proof of the general Cauchy formula.

The following result will establish that if f is analytic on a closed disk, then the value of f at any interior point is completely determined by its values on the boundary, and furthermore, there is an explicit formula describing the dependence.

Theorem 2.2.7 (Cauchy's Integral Formula for a Circle). *Let f be analytic on an open set U containing the circle $\Gamma = \{|z - z_0| = r\}$ and its interior V . Then for any $z \in V$,*

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(w)}{w - z} dw$$

Proof. Fix $z \in V$ and let $g : U \rightarrow \mathbf{C}$ be defined by

$$g(w) = \begin{cases} \frac{f(w) - f(z)}{w - z} & \text{if } w \neq z \\ f'(z) & \text{otherwise.} \end{cases}$$

Then g is continuous on U and analytic on $U \setminus \{z\}$. Hence, by Theorem 2.2.6 (note that it is possible to find a larger disk $D \subset U$ containing \bar{V} , and that D is convex) $\int_{\Gamma} g(w) dw = 0$, that is,

$$\frac{1}{2\pi i} \int_{\Gamma} \frac{f(w)}{w - z} dw = \frac{f(z)}{2\pi i} \int_{\Gamma} \frac{1}{w - z} dw.$$

Now

$$\begin{aligned} \int_{\Gamma} \frac{1}{w - z} dw &= \int_{\Gamma} \frac{1}{w - z_0 - (z - z_0)} dw \\ &= \int_{\Gamma} \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(w - z_0)^{n+1}} dw. \end{aligned}$$

Since

$$\left| \frac{(z - z_0)^n}{(w - z_0)^{n+1}} \right| = \frac{|z - z_0|^n}{r^{n+1}}$$

and $|z - z_0| < r$, the series inside the integral converges uniformly and can be integrated term by term. Now

$$\int_{\Gamma} \frac{1}{(w - z_0)^{n+1}} = \begin{cases} 0 & \text{if } n = 1, 2, \dots \\ 2\pi i & \text{if } n = 0 \end{cases}$$

Therefore

$$\int_{\Gamma} \frac{1}{w - z} dw = 2\pi i.$$

□

The preceding proof evaluates a useful contour integral.

Corollary 2.2.8. *If Γ is a circle, then*

$$\int_{\Gamma} \frac{1}{w - z} dw = \begin{cases} 2\pi i & \text{if } z \text{ is inside } \Gamma \\ 0 & \text{if } z \text{ is outside } \Gamma \end{cases}$$

Proof. If z is inside Γ , the result follows from Theorem 2.2.7. If z is outside Γ , then $f(w) = 1/(w - z)$ is analytic on an open disk containing Γ but not on z , hence Cauchy's Theorem for a Convex Set (Theorem 2.2.6) applies. □

Theorem 2.2.9 (Cauchy's Formula for Derivatives). *Under the hypothesis of Theorem 2.2.7, f has derivatives of all orders at $z \in U$, and the n th derivative $f^{(n)}$ is given by*

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\Gamma} \frac{f(w)}{(w - z)^{n+1}} dw.$$

Proof. First take $n = 1$. Then by Theorem 2.2.7 (Cauchy's Theorem for a Circle),

$$\begin{aligned} & \frac{f(z+h) - f(z)}{h} - \frac{1}{2\pi i} \int_{\Gamma} \frac{f(w)}{(w - z)^2} dz \\ &= \frac{1}{2\pi i h} \int_{\Gamma} \left[\frac{f(w)}{w - z - h} - \frac{f(w)}{w - z} - \frac{hf(w)}{(w - z)^2} \right] dw \\ &= \frac{h}{2\pi i} \int_{\Gamma} \frac{f(w)}{(w - z)^2 (w - z - h)} dw \end{aligned}$$

which converges to 0 as $h \rightarrow 0$ because f is bounded on Γ and $|(w - z)(w - z - h)|$ is bounded below away from 0 for $w \in \Gamma$, z a fixed point in V , and h sufficiently small in absolute value so that the closed disk $\bar{D}(z; |h|)$ is contained in V .

Assume that the result has been proved for $n = k$. Then, as above

$$\begin{aligned} & \frac{f^{(k)}(z+h) - f^{(k)}(z)}{h} - \frac{(k+1)!}{2\pi i} \int_{\Gamma} \frac{f(w)}{(w - z)^{k+2}} dz \\ &= \frac{k!}{2\pi i h} \int_{\Gamma} \left[\frac{f(w)}{(w - z - h)^{k+1}} - \frac{f(w)}{(w - z)^{k+1}} - \frac{h(k+1)f(w)}{(w - z)^{k+2}} \right] dw \end{aligned}$$

The expression in brackets is of the order of h^2 as $h \rightarrow 0$ (form a common denominator and use the binomial theorem) so that the integral approaches 0 as $h \rightarrow 0$. □

Corollary 2.2.10. *If f is analytic on the open set U , then f has derivatives of all orders on U . In particular, if f has a primitive, then f is analytic on U .*

Proof. This is immediate from Theorems 2.2.7 and 2.2.9. In particular, if F is a primitive for f , then $F'' = f'$ exists. \square

The following theorem takes care of the loose end in the statement of Theorem 2.2.6.

Theorem 2.2.11 (Removable Singularity). *Let U be an open set and $f : U \rightarrow \mathbf{C}$ be continuous. If $z_0 \in U$ and f is analytic on $U \setminus \{z_0\}$, then f is analytic on U .*

Proof. Let $D = D(z_0; r) \subset U$; it suffices to show that f is analytic on D . By Theorem 2.2.6, $\int_{\gamma} f = 0$ for every closed path γ in D , and thus f has a primitive on D , because of Theorem 2.2.4. By Corollary 2.2.10, f is analytic on D . \square

Theorem 2.2.12 (Morera's theorem). *Let U be an open subset of \mathbf{C} and let $f : U \rightarrow \mathbf{C}$ be continuous. If $\int_{\partial\Delta} f = 0$ for every triangle $\Delta \subset U$, then f is analytic on U .*

Proof. Let D be a disk contained in U . By Theorem 2.2.5 f has a primitive on D ; by Corollary 2.2.10, f is analytic on D . \square

Theorem 2.2.13. *Let γ be a path, and let $g : \gamma^* \rightarrow \mathbf{C}$ be continuous. Define, for each $z \in \mathbf{C} \setminus \gamma^*$,*

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{g(w)}{w-z} dw.$$

Then f is differentiable at z and

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\gamma} \frac{g(w)}{(w-z)^{n+1}} dw.$$

Thus f and all its derivatives are analytic on $\mathbf{C} \setminus \gamma^$. Furthermore,*

$$f^{(n)}(z) \rightarrow 0$$

as $z \rightarrow \infty$ for each n .

Proof. Because γ^* is compact, it is contained in a closed disc $|z| \leq r$. Because g is continuous on γ^* , there is a constant M such that $|g| \leq M$ on γ^* . Then, for $|z| > r$,

$$|f^{(n)}(z)| \leq \frac{n!}{2\pi} \frac{M}{(|z|-r)^{n+1}} (\text{length } \gamma),$$

which converges to 0 as $z \rightarrow \infty$. \square

The Cauchy integral formula allows to show that analytic functions can be locally represented by power series.

Theorem 2.2.14 (Taylor Series). *Let f be analytic on $D(z_0; r)$. Then*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n,$$

for $z \in D(z_0; r)$. The series converges absolutely on $D(z_0; r)$, and uniformly on compact subsets of this disk.

Proof. Let $|z - z_0| < r_1 < r$, and let Γ be the circle $|z - z_0| = r_1$. By Theorem 2.2.7 (Cauchy's Integral Formula for a Circle),

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{\Gamma} \frac{f(w)}{w - z} dw \\ &= \frac{1}{2\pi i} \int_{\Gamma} \frac{f(w)}{w - z_0} \left[1 - \frac{z - z_0}{w - z_0} \right]^{-1} dw \\ &= \frac{1}{2\pi i} \int_{\Gamma} \sum_{n=0}^{\infty} f(w) \frac{(z - z_0)^n}{(w - z_0)^{n+1}} dw \end{aligned}$$

Now, if $|z - z_0| \leq r_2 < r_1$,

$$\frac{|f(w)| |(z - z_0)^n|}{|w - z_0|^{n+1}} \leq \frac{M(f; \Gamma) \left(\frac{r_1}{r_2}\right)^n}{r_1}$$

hence the series

$$\sum_{n=0}^{\infty} f(w) \frac{(z - z_0)^n}{(w - z_0)^{n+1}}$$

converges uniformly for $w \in \Gamma$ and can therefore be integrated term by term. We obtain

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} (z - z_0)^n \frac{1}{2\pi i} \int_{\Gamma} \frac{f(w)}{(w - z_0)^{n+1}} dw \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n \end{aligned}$$

by Theorem 2.2.9. The statements about absolute and uniform convergence follow from Theorem 1.5.4. \square

In order to prove the converse to Theorem 2.2.14 (Taylor series), namely that a convergent power series defines an analytic function, the following basic result is required.

Theorem 2.2.15 (Analytic Limit). *Let f_1, f_2, \dots be analytic on the open set $U \subset \mathbf{C}$, and assume that $f_n \rightarrow f$ on U , uniformly on compact subsets. Then f is analytic on U . Furthermore, for each k , $f_n^{(k)} \rightarrow f^{(k)}$ on U , uniformly on compact subsets.*

Proof. Since $f_n \rightarrow f$ uniformly on compact subsets, f is continuous. Let Δ be a triangle in U . Since Δ is compact,

$$\lim_{n \rightarrow \infty} \int_{\partial \Delta} f_n = \int_{\partial \Delta} f$$

Each $\int_{\partial \Delta} f_n = 0$, by Cauchy's Theorem for a convex set. Thus $\int_{\partial \Delta} f = 0$, and f is analytic, by Morera's theorem.

Let $D(z_0; r)$ be a disk whose closure is contained in U , and let Γ be the circle $|w - z_0| = r$. If $z \in D(z_0; r)$, then Theorem 2.2.7 (Cauchy's Integral Formula for a Circle) yields

$$f_n(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f_n(w)}{w - z} dw.$$

Then $f_n(w)/(w - z) \rightarrow f(w)/(w - z)$ uniformly on Γ , because Γ is compact and $|w - z|$ is bounded away from 0 for $w \in \Gamma$. Thus

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(w)}{w - z} dw$$

Now f is analytic at z because of Theorem 2.2.13, and since z was an arbitrary point of U , f is analytic on U .

By Cauchy's Formula for the Derivatives,

$$f_n^{(k)}(z) = \frac{k!}{2\pi i} \int_{\Gamma} \frac{f_n(w)}{(w-z)^{k+1}} dw$$

for $z \in D(z_0; r)$, and similarly for f . Therefore,

$$f_n^{(k)}(z) - f^{(k)}(z) = \frac{k!}{2\pi i} \int_{\Gamma} \frac{f_n(w) - f(w)}{(w-z)^{k+1}} dw$$

If $|z - z_0| < r_1 < r$, then by Lemma 2.1.4 (Basic Integral Estimate),

$$|f_n^{(k)}(z) - f^{(k)}(z)| \leq \frac{k!}{2\pi i} \max_{w \in \Gamma} |f_n(w) - f(w)| \frac{2\pi r}{(r-r_1)^{k+1}} \rightarrow 0$$

as $n \rightarrow \infty$. Therefore, $f_n^{(k)} \rightarrow f^{(k)}$ uniformly on any closed subdisk of $D(z_0; r)$, and since $z_0 \in U$ was arbitrary, uniformly on compact subsets of U . \square

The main result of this section is the following.

Theorem 2.2.16. (a) *If f is analytic at x_0 , then f is representable by a power series in a neighborhood of z_0 .*

(b) *If f can be represented by a power series expansion $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ in a neighborhood of z_0 , then f is analytic at z_0 .*

Proof. Part (a) follows from Theorem 2.2.14 (Taylor Series). To prove (b), note that each finite sum $\sum_{n=0}^k a_n(z - z_0)^n$ is analytic on \mathbf{C} , and that by hypothesis converges to f on some disc $D(z_0; r)$, uniformly on compact subsets. Thus f is analytic on $D(z_0; r)$ by Theorem 2.2.15 (Analytic Limit). \square

Corollary 2.2.17. (a) *If $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$, $z \in D(z_0; r)$, then*

$$f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)a_n(z - z_0)^{n-k}$$

for $k = 1, 2, \dots$.

(b) *Any power series representation of f in a neighborhood of z_0 must coincide with the Taylor expansion; that is, if*

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$$

on $D(z_0; r)$, then

$$a_k = \frac{1}{k!} f^{(k)}(z_0)$$

for all k .

Proof. The sequence $\sum_{n=0}^k a_n(z - z_0)^n \rightarrow f(z)$ uniformly on compact subsets of $D(z_0; r)$. Therefore, by Theorem 2.2.15 (Analytic Limit), \square

Exercise 2.2.18. If f is analytic at z_0 , then show that it is not possible that $|f^{(k)}| \geq k!b_k$ for all $k = 1, 2, \dots$, where $(b_k)^{1/k} \rightarrow \infty$ as $k \rightarrow \infty$.

Solution. If so, then the coefficients of the Taylor series of f at z_0 would satisfy $\limsup_k a_k^{1/k} = \infty$, hence the radius of convergence of $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ would be 0, a contradiction. \square

2.3 Applications

Theorem 2.3.1 (Cauchy's Estimate). *Let f be analytic on $D(z_0; R)$. Then, for $0 < r < R$,*

$$|f^{(n)}(z_0)| \leq \frac{n!}{r^n} \sup\{|f(z)| \mid |z - z_0| = r\}$$

Thus, if $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$, $z \in D(z_0; R)$, then

$$|a_n| \leq \frac{1}{r^n} \sup\{|f(z)| \mid |z - z_0| = r\}$$

Proof. Apply Theorem 2.2.7 (Cauchy's Integral Formula for a Circle) to $f^{(n)}$ and then use Lemma 2.1.4 (Basic Integral Estimate). \square

Definition 2.3.2. A function which is analytic on the whole plane \mathbf{C} is called an *entire* function.

Theorem 2.3.3 (Liouville's Theorem). *If f is a bounded entire function, then f is constant.*

Proof. Expand f in a Taylor series about $z_0 = 0$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$, and apply Theorem 2.3.1 (Cauchy's Estimate) to obtain

$$|a_n| \leq \frac{1}{r^n} \sup\{|f(z)| \mid |z - z_0| = r\}.$$

If $n \geq 1$, then the right side converges to 0 as $r \rightarrow \infty$ because f is bounded. Therefore $f \equiv a_0$ is constant. \square

Theorem 2.3.4 (Fundamental Theorem of Algebra). *If $P(z)$ is a polynomial in z of degree at least 1, then $P(z) = 0$ for some $z \in \mathbf{C}$.*

Proof. If P is never 0, then $1/P$ is entire. Since $|P(z)| \rightarrow \infty$ as $|z| \rightarrow \infty$, $1/P$ is bounded, hence constant by Theorem 2.3.3 (Liouville's Theorem). \square

Exercise 2.3.5. Show that if $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ is a polynomial of degree $n \geq 1$, then $|P(z)| \rightarrow \infty$ as $|z| \rightarrow \infty$. In fact, show that if $|z| \geq \max\{1, 2n|a_{n-1}|, \dots, 2n|a_0|\}$, then $|P(z)| \geq |z|^n/2$.

Solution. Write

$$P(z) = z^n \left(1 + \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n} \right),$$

and let

$$M = \max\{1, 2n|a_{n-1}|, \dots, 2n|a_0|\}$$

If $|z| \geq M$, then $|z|^k \geq |z| \geq M$ for all k , and

$$\frac{|a_{n-k}|}{|z^k|} \leq \frac{|a_{n-k}|}{|z|} \leq \frac{|a_{n-k}|}{2n|a_{n-k}|} = \frac{1}{2n}.$$

Therefore

$$\left| \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n} \right| \leq n \frac{1}{n} = \frac{1}{2}$$

and so

$$\begin{aligned} |P(z)| &\geq |z^n| \left| 1 + \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n} \right| \\ &\geq |z^n| \left(1 - \left| \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n} \right| \right) \\ &\geq \frac{|z^n|}{2}. \end{aligned}$$

□

If f is analytic on an open set containing a circle Γ and its interior U , then the values of f on Γ determine its values on U . We will show that if S is a subset of U which has a limit point in U , then the values of f on S determine the values of f on U .

Definition 2.3.6. Let f be analytic at z_0 with power series representation $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$. Then f is said to have a *zero of order m* at z_0 if $a_n = 0$ for $n < m$ and $a_m \neq 0$.

This means that $f(z) = (z - z_0)^m g(z)$, where g is analytic at z_0 and $g(z_0) \neq 0$. A zero of order 1 is also called a *simple zero*.

Proposition 2.3.7. Let f be analytic on the connected open set U . Suppose that the set of zeros of f on U has a limit point in U . Then f is identically zero on U .

Proof. Let A be the set of points of U which are limit points of zeros of f . This is a closed subset of U because f is continuous. We will show that either A is empty or else $A = U$.

Suppose that there exists $z_0 \in A$. Expand f in a Taylor series about z_0 , $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$. Suppose that not all $a_n = 0$, and let m be the least integer such that $a_m \neq 0$. Then $f(z) = (z - z_0)^m g(z)$, where $g(z_0) \neq 0$. By continuity, $g(z) \neq 0$ in a neighborhood of z_0 , contradicting that z_0 is a limit point of zeros of f . Thus if z_0 is a limit point of zeros of f , then f is identically zero in a neighborhood of z_0 . Thus, if $z_0 \in A$, there is a neighborhood of z_0 also contained in A . □

Theorem 2.3.8 (Identity Theorem). Let f and g be analytic on the connected open set U . If $f = g$ on a subset S of U which has a limit point in U , then $f = g$ on U .

Proof. Apply the proposition to the function $f - g$. □

Example 2.3.9. The power series

$$1 + \frac{1}{1!}z + \frac{1}{2!}z^2 + \cdots$$

defines an analytic function $f(z)$ in \mathbf{C} . Previously, we have defined the exponential function e^z , $z = x + iy$, by

$$e^z = e^x(\cos y + i \sin y).$$

If z is real, then $y = 0$, and we know from calculus that the exponential function e^x is given by the power series $\sum_{n=0}^{\infty} x^n/n!$. Therefore, the identity theorem guarantees that

$$e^z = 1 + \frac{1}{1!}z + \frac{1}{2!}z^2 + \cdots$$

Proposition 2.3.10. *If f is analytic on an open set containing the disk $|z - z_0| \leq r$, then*

$$|f(z_0)| \leq \sup\{|f(z)| \mid |z - z_0| = r\}$$

Proof. This is an immediate corollary of Cauchy's Theorem for a Disk (Theorem 2.2.7) and the Basic Integral Estimate. \square

Theorem 2.3.11 (Maximum Principle). *Let f be analytic and not identically constant on the connected open set $U \subset \mathbf{C}$.*

(a) *If $D(z_0; r) \subset U$, then there exists $z_1 \in D(z_0; r)$ such that $|f(z_0)| < |f(z_1)|$.*

(b) *If $M = \sup_{z \in U} |f(z)|$, then $|f(z)| < M$, for all $z \in U$.*

(c) *If U is bounded and ∂U is the boundary of U , then*

$$|f(z)| < \limsup_{w \rightarrow \partial U} |f(w)|,$$

for all $z \in U$.

(d) *If U is bounded and f is defined and continuous on the closure of U , then $|f(z)| < \max_{\partial U} |f|$.*

Proof. (a) If it is false, then $|f(z_0 + re^{it})| \leq |f(z_0)|$ for all sufficiently small r . Theorem ?? implies

$$\begin{aligned} |f(z_0)| &\leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + re^{it})| dt \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0)| dt \\ &= |f(z_0)| \end{aligned}$$

Thus

$$\int_0^{2\pi} (|f(z_0)| - |f(z_0 + re^{it})|) dt = 0$$

for all small r . Since the integrand is continuous and positive, it must be identically 0. Therefore, there is a neighborhood of z_0 where $|f|$ is constant. But then f itself is constant on this neighborhood.

(a) implies (b): if $|f(z_0)| = M$ for some $z_0 \in U$, then by (a) there exists z_1 in U satisfying $|f(z_1)| > |f(z_0)|$, contradicting the definition of M .

(b) implies (c): let z_n be a sequence in U such that $|f(z_n)| \rightarrow M$. Since U is bounded, the closure of U is compact; therefore, after passing to a subsequence if necessary, there is $z_0 \in \bar{U}$ such that $z_n \rightarrow z_0$. If $z_0 \in U$, then $|f(z_0)| = M$ by continuity of f , contradicting (b); therefore $z_0 \in \partial U$. Thus $M \leq M_1$ and $|f(z)| < M_1$ by (b).

(c) implies (d): because f is continuous on U , if z_n is a sequence in U converging to a point z_0 in ∂U , then $f(z_n) \rightarrow f(z_0)$. \square

The absolute value of a function may have a local minimum; for example at a point where it vanishes. This is the only case in which it can happen.

Theorem 2.3.12 (Minimum principle). *Let f be analytic on the connected open set U and assume that $f(z) \neq 0$ for all $z \in U$.*

- (a) If $D(z_0; r) \subset U$, then there exists $z_1 \in D(z_0; r)$ such that $|f(z_1)| < |f(z_0)|$.
- (b) If $m = \inf_{z \in U} |f(z)|$, then $|f(z)| > m$, for all $z \in U$.
- (c) If U is bounded and ∂U is the boundary of U , then

$$|f(z)| > \liminf_{w \rightarrow \partial U} |f(w)|,$$

for all $z \in U$.

- (d) If U is bounded and f is defined and continuous on the closure of U , then $|f(z)| > \min_{\partial U} |f|$ for all $z \in U$.

Proof. Apply Theorem 2.3.11 (Maximum Principle) to the function $1/f$. □

There are also maximum and minimum principles for the real and imaginary parts of an analytic function.

Theorem 2.3.13. *Let f be analytic and not identically constant on the open connected set U . If $f = u + iv$, then u and v satisfy Theorem 2.3.11 and Theorem 2.3.12 on U .*

Proof. The function $\exp f$ is analytic, never 0, and not constant on U . Since $|\exp f| = \exp u$, the function u satisfies both the Maximum and Minimum principles on U . The same observation applies to $\exp -if$, whose absolute value is $|\exp -if| = \exp v$. □

Chapter 3

The Cauchy Formula

3.1 Logarithms

The exponential function was described in Section 1.6. This is an analytic mapping $\exp : \mathbf{C} \rightarrow \mathbf{C}$ whose image is the plane minus the origin. A branch of the logarithm on an open subset U of $\mathbf{C} \setminus \{0\}$ is a map $h : U \rightarrow \mathbf{C}$ such that $\exp h(z) = z$. The elementary branches of the logarithm are of the form $\log z = \ln |z| + i\theta(z)$, where $\theta(z)$ is the argument of z in $(\theta_0, \theta_0 + 2\pi]$.

In general, there are analytic functions which are branches of the logarithm but do not coincide with one of the elementary branches. For example, let U be the open set depicted in the figure and define

$$h(z) = \begin{cases} \ln |z| + i\theta(z), & 0 \leq \theta < 2\pi, \text{ for } z \in U_1 \\ \ln |z| + i\theta(z), & \pi \leq \theta < 3\pi, \text{ for } z \in U_2 \end{cases}$$

Locally h coincides with one of the elementary branches of $\log z$, so it is an analytic branch of \log on U . However, a single branch will not suffice since any elementary branch will be discontinuous on a ray through the origin, and any such ray intersects U .

Definition 3.1.1. Let $S \subset \mathbf{C}$ and $f : S \rightarrow \mathbf{C} \setminus \{0\}$ be continuous. The function $g : S \rightarrow \mathbf{C}$ is a *continuous logarithm* of f if g is continuous on S and $\exp g(z) = f(z)$ for all $z \in S$. The function $\theta : S \rightarrow \mathbf{R}$ is a *continuous argument* of f if θ is continuous on S and $f(z) = |f(z)| \exp(i\theta(z))$ for all $z \in S$.

Continuous logarithms are closely related to continuous arguments.

Theorem 3.1.2. Let $S \subset \mathbf{C}$ and $f : S \rightarrow \mathbf{C} \setminus \{0\}$ be continuous.

- (a) If g is a continuous logarithm of f , then $\Im g$ is a continuous argument of f .
- (b) If θ is a continuous argument of f , then $\ln |f(z)| + i\theta(z)$ is a continuous logarithm.
- (c) If S is connected and g_1, g_2 are continuous logarithms of f , then $g_1 - g_2 = 2\pi in$ for some integer n ; if θ_1, θ_2 are continuous arguments of f , then $\theta_1 - \theta_2 = 2\pi m$ for some integer m .
- (d) If S is connected and $z, w \in S$, then

$$g(w) - g(z) = \ln |f(w)| - \ln |f(z)| + i\theta(w) - i\theta(z)$$

for all continuous logarithms g and continuous arguments θ of f .

Proof. (a) $f = \exp g = \exp(\Re g + i\Im g) = |f| \exp \Im g$.

(b) $\exp(\ln |f| + i\theta) = |f|e^{i\theta} = f$.

It follows that a non vanishing continuous function on a set S has a continuous logarithm if and only if it has a continuous argument.

(c) If $e^{g_1} = e^{g_2} = f$, then $g_1 - g_2$ is an integer multiple of $2\pi i$. Since S is connected, this multiple is the same in all of S . Similarly if $e^{i\theta_1} = e^{i\theta_2} = f/|f|$.

(d) By (b), $\ln |f| + i\theta$ is a continuous logarithm of f . By (c), $g = \ln |f| + i\theta + 2\pi in$ for some integer n , and the result follows. \square

In general, a given function may not have a continuous logarithm.

Exercise 3.1.3. Let $S = \{|z| = 1\}$ and $f(z) = z$. Show that f does not have a continuous argument function on S .

Solution. Let θ be a continuous argument of f on S , so that $z = e^{i\theta(z)}$ for $|z| = 1$. Let $\gamma(t) = e^{it}$, $0 \leq t \leq 2\pi$. Then $e^{it} = \exp i\theta(e^{it})$. Then t and $\theta(e^{it})$ are both continuous arguments of γ , hence $\theta(e^{it}) = t + 2\pi k$ for some integer k . Let $t \rightarrow 0$ to obtain $\theta(1) = 2\pi k$ and let $t \rightarrow 1$ to obtain $\theta(1) = 1 + 2\pi k$, and contradiction. \square

However, if S is a closed interval $[a, b]$, then f will have a continuous logarithm.

Theorem 3.1.4. Let $f : [a, b] \rightarrow \mathbf{C} \setminus \{0\}$ be continuous. Then f has a continuous logarithm on S .

Proof. Suppose that the image of f is contained in a disk D which does not contain 0. Then there is an analytic branch of the logarithm function $\log z$ defined on D , that is, there is a function h analytic on D such that $\exp h(z) = z$. Therefore, $g(t) = h(f(t))$ is an analytic logarithm of f .

In general, cover the image of f by disks like above, and patch up logarithms. This is done as follows. Since $[a, b]$ is compact, $|f|$ has a minimum $m > 0$ on $[a, b]$. Form a partition $a = t_0 < \dots < t_n = b$ such that $|f(t) - f(t_j)| < m$ on $[t_j, t_{j+1}]$ for all j . Then $f(t) \in D = D(f(t_j), m)$ for t in $[t_j, t_{j+1}]$, and $0 \notin D$ by definition of m . With h as above we have $e^{h(f(t))} = f(t)$ on t_j, t_{j+1} . If $e^{g_0(t)} = f(t)$ for $t_0 \leq t \leq t_1$ and $e^{g_2(t)} = f(t)$ for $t_1 \leq t \leq t_2$, then $g_0(t_1) = g_1(t_1) + 2\pi in$. Replace g_1 by $g_1 + 2\pi in$ on $[t_1, t_2]$. This produces a continuous logarithm of f on $[t_0, t_1]$. Repeat the process inductively to obtain a continuous logarithm of f on $[a, b]$. \square

Definition 3.1.5. Let f be analytic and never 0 on the open set U . A function $g : U \rightarrow \mathbf{C}$ is an *analytic logarithm* of f on U if g is analytic on U and $\exp g(z) = f(z)$ for all z in U .

Theorem 3.1.6. Let f be analytic and never 0 on U . Then f has an analytic logarithm on U if and only if the function f'/f has a primitive on U .

Proof. If $\exp g = f$ on U , then the chain rule implies $g' \exp g = f'$, and hence $f'/f = g'$.

Conversely, if g is analytic on U and $g' = f'/f$, then $f \exp(-g)$ has zero derivative on U and thus $f \exp(-g)$ is equal to a constant k_A on each component A of U . Choose c_A so that $e^{c_A} = k_A$ (the constant $k_A \neq 0$). Then $\exp(g + c_A) = f$ on A . \square

Analytic logarithms exists under fairly general conditions.

Theorem 3.1.7. Let $U \subset \mathbf{C}$ be a convex open set, and let f be analytic and never 0 on U . Then f has an analytic logarithm on U . More generally, if U is an open subset such that $\int_\gamma h = 0$ for every closed path in U and every analytic function h , then every analytic function f which is never 0 on U has an analytic logarithm on U .

Proof. First consider the convex case. If f'/f is analytic on U , then Cauchy's Theorem for a Convex Set (Theorem 2.2.6) yields $\int_{\gamma} f'/f = 0$ for every closed path γ in U . By Theorem 3.1.6, f has an analytic logarithm. \square

Theorem 3.1.8. *If g is an analytic logarithm of f on the open set U , and γ is a path in U , then*

$$\int_{\gamma} \frac{f'(z)}{f(z)} dz = g(\gamma(b)) - g(\gamma(a))$$

where $[a, b]$ is the parameter interval of γ .

Proof. By the argument of Theorem 3.1.6, $g' = f'/f$ on U . The result follows at once. \square

3.2 The Index of a point with respect to a curve

Definition 3.2.1. If X is a topological space, a *curve in X* is a continuous mapping γ of a compact interval $[a, b]$ in \mathbf{R} into X , here $a < b$. We call $[a, b]$ the *parameter interval* of γ and denote the range of γ by γ^* . Thus γ is a mapping, and γ^* is the set of all points of the form $\gamma(t)$, for $a \leq t \leq b$. We say that γ is a closed curve if $\gamma(a) = \gamma(b)$.

Theorem 3.2.2. *Let γ be a closed curve such that $0 \notin \gamma^*$. Let θ be a continuous argument of γ . Then*

$$\frac{1}{2\pi}(\theta(b) - \theta(a))$$

is an integer, independent of the continuous argument chosen.

Proof. For each $t \in [a, b]$, $e^{i\theta(t)} = \gamma(t)/|\gamma(t)|$. Thus

$$e^{i\theta(b) - i\theta(a)} = \frac{\gamma(b)}{|\gamma(b)|} \frac{|\gamma(a)|}{\gamma(a)} = 1,$$

since γ is closed. It follows that $\theta(b) - \theta(a)/2\pi$ is an integer.

If ϕ is another continuous argument, then $\phi(t) = \theta(t) + 2\pi m$ for some integer m , and the result follows. \square

Definition 3.2.3. Let γ be a closed curve, and z_0 a point not in γ^* . Let θ be a continuous argument of the curve $\gamma - z_0$. The index of z_0 with respect to γ is the integer

$$\text{ind}(z_0; \gamma) = \frac{\theta(b) - \theta(a)}{2\pi}.$$

Intuitively, $\text{ind}(\gamma; z_0)$ is the net number of revolutions of $\gamma(t)$ about z_0 , so sometimes it is called the winding number.

Lemma 3.2.4. *Let γ be a curve, $V \subset \mathbf{C}$ an open subset which contains γ^* . Then there is a partition $a = t_0 < \dots < t_n = b$ and disks $D_1, \dots, D_n \subset V$ such that $\gamma[t_{j-1}, t_j] \subset D_j$, for all $j = 1, \dots, n$.*

Proof. Let $\varepsilon = \text{dist}(\gamma^*, \mathbf{C} \setminus V) > 0$. By uniform continuity of γ , there is a $\delta > 0$ such that $|t - t'| < \delta$ implies $|\gamma(t) - \gamma(t')| < \varepsilon$. Let $a = t_0 < t_1 < \dots < t_n = b$ be a partition of $[a, b]$ such that $|t_j - t_{j-1}| < \delta$ for all j . Let $D_j = D(\gamma(t_j); \varepsilon) \subset V$, $j = 1, \dots, n$. If $t_{j-1} \leq t \leq t_j$, then $|t - t_j| < \delta$ and $|\gamma(t) - \gamma(t_j)| < \varepsilon$, and thus $\gamma(t) \in D_j$. \square

Theorem 3.2.5 (Index of a Point with Respect to a Path). *Let γ be a closed path, z_0 a point not in γ^* . Then*

$$\text{ind}(\gamma; z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{1}{z - z_0} dz$$

More generally, if f is analytic on γ^ and $z_0 \notin (f \circ \gamma)^*$, then*

$$\text{ind}(f \circ \gamma; z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z) - z_0} dz$$

Proof. Because γ^* is compact, there is an open set V containing γ^* such that $f - z_0$ is never 0 on V . (Indeed, because the set $(f \circ \gamma)^*$ is compact and does not contain z_0 , there is an open set U which contains $(f \circ \gamma)^*$ but not z_0 . It suffices to take an open set $V \subset f^{-1}(U)$ containing γ^* where f is defined.) Construct a partition $a = t_0 < \dots < t_n = b$ and disks $D_1, \dots, D_n \subset V$ as in Lemma 3.2.4. Then $f - z_0$ has an analytic logarithm g_j on each D_j , and so, if γ_j is the restriction of γ to $[t_{j-1}, t_j]$, then

$$\int_{\gamma_j} \frac{f'(z)}{f(z) - z_0} dz = g_j(\gamma(t_j)) - g_j(\gamma(t_{j-1})).$$

Let $\theta(t)$ be a continuous argument $f(\gamma(t)) - z_0$ on $[a, b]$. Because $g_j(\gamma(t))$ is a continuous logarithm of $f(\gamma(t)) - z_0$ on $[t_{j-1}, t_j]$, it follows that

$$\begin{aligned} \frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z) - z_0} dz &= \frac{1}{2\pi i} \sum_{j=1}^n g_j(\gamma(t_j)) - g_j(\gamma(t_{j-1})) \\ &= \frac{1}{2\pi i} \sum_{j=1}^n \left[\ln |f(\gamma(t_j)) - z_0| - \ln |f(\gamma(t_{j-1})) - z_0| \right. \\ &\quad \left. + i\theta(t_j) - i\theta(t_{j-1}) \right] \\ &= \frac{\theta(b) - \theta(a)}{2\pi} \\ &= \text{ind}(f \circ \gamma; z_0) \end{aligned}$$

□

Finally we consider the behavior of $\text{ind}(\gamma; z)$ as z varies.

Theorem 3.2.6. *If γ is a closed path, then $\text{ind}(\gamma; z)$, regarded as a function of z , is constant on each component of $\mathbf{C} \setminus \gamma^*$, and equal to 0 on the unbounded component.*

Proof. The function $\text{ind}(\gamma; \bullet)$ is analytic, hence continuous on $\mathbf{C} \setminus \gamma^*$. But on any given component of $\mathbf{C} \setminus \gamma^*$, $\text{ind}(\gamma; \bullet)$ is a continuous, integer valued function, and is therefore constant.

Since γ^* is compact, it lies on a bounded disk D whose complement D^c is connected; thus D^c lies in some component of $\mathbf{C} \setminus \gamma^*$. This shows that $\mathbf{C} \setminus \gamma^*$ has precisely one unbounded component.

By Theorem 2.2.13, $\text{ind}(\gamma; z) \rightarrow 0$ as $z \rightarrow \infty$, hence $\text{ind}(\gamma; \bullet)$ must be 0 on the unbounded component of $\mathbf{C} \setminus \gamma^*$. □

3.3 Cauchy's Theorem

The statement of the general form of Cauchy's Theorem requires to integrate over objects slightly more general than closed paths.

Definition 3.3.1. A *cycle* is a formal sum $\sigma = n_1\gamma_1 + \cdots + n_k\gamma_k$, where the a_i are integers and the γ_i are closed paths. The union $\bigcup_{j=1}^k \gamma_j^*$ of the images of the individual paths forming σ is denoted by σ^* .

If f is a continuous function on γ^* , then define

$$\int_{\gamma} f = \sum_{j=1}^k a_j \int_{\gamma_j} f.$$

A cycle γ is said to be *equivalent to 0* if

$$\int_{\gamma} f = 0$$

for all continuous functions f on γ^* . Two cycles γ_1 and γ_2 are *equivalent* if the difference $\gamma_1 - \gamma_2$ is equivalent to 0.

Finally, the index of a point z with respect to a cycle $\gamma = \sum_{j=1}^n n_j\gamma_j$ is defined to be

$$\text{ind}(\gamma; z) = \sum_{j=1}^k n_j \text{ind}(\gamma_j; z).$$

If $U \subset \mathbf{C}$ is an open set, and γ is a cycle with $\gamma^* \subset U$, then γ is called a cycle in U . Cycles can be added in the obvious way, thus the space of cycles forms a commutative group, denoted by $Z(U)$.

Definition 3.3.2. A cycle σ on U is homologous to 0 on U , written $\sigma \sim 0 \pmod{U}$ if $\text{ind}(\sigma; z) = 0$ for all z not in U .

It is clear that the space of cycles in U which are homologous to 0 modulo U is a subgroup of the group of cycles in U . This subgroup is denoted by $B(U)$, and the quotient group $Z(U)/B(U) = H(U)$ is called the first homology group of U with integer coefficients.

Theorem 3.3.3. Let U be a connected open subset of \mathbf{P} . Then $H_1(U)$ is a free commutative group of rank equal to the number of components of the complement of U in \mathbf{P} minus one.

The following technical lemma will be very useful for subsequent developments.

Lemma 3.3.4 (Fundamental Lemma). Let γ be a closed polygonal path whose edges are parallel to the coordinate axis. For a mesh consisting of all lines parallel to the axes which passes through the vertices of γ . The complement of this mesh consists of regions of three types: bounded rectangular regions R_1, \dots, R_k , unbounded regions having three sides, and unbounded regions having two sides. Choose points $z_j \in R_j$, and let γ_0 be the cycle

$$\gamma_0 = \sum_{j=1}^k \text{ind}(\gamma; z_j) \partial R_j$$

where ∂R_j is the closed path determined by the boundary of R_j , oriented counterclockwise. Then γ and γ_0 are equivalent.

Proof. First note that

$$(1) \quad \text{ind}(\gamma_0; z_k) = \sum_{j=1}^m \text{ind}(\gamma; z_j) \text{ind}(\partial R_j; z_k) = \text{ind}(\gamma; z_k),$$

for $k = 1, \dots, m$. Also, if $z'_k \in R'_k$,

$$\begin{aligned} \text{ind}(\gamma_0; z'_k) &= \sum_{j=1}^m \text{ind}(\gamma; z_j) \text{ind}(\partial R_j; z'_k) \\ &= 0 \\ &= \text{ind}(\gamma; z'_k) \end{aligned}$$

because z'_k belongs to the unbounded component of $\mathbf{C} \setminus \gamma^*$.

Now suppose that σ_{ij} is an edge lying between R_i and R_j . Suppose that in $\gamma - \gamma_0$, the edge σ_{ij} traverses c times (c is an integer, possibly negative). Let $\sigma = \gamma - \gamma_0 - c\partial R_i$. Then

$$(3) \quad \text{ind}(\sigma; z_i) = \text{ind}(\gamma; z_i) - \text{ind}(\gamma_0; z_i) - c \text{ind}(\partial R_i; z_i) = -c$$

by (1). Also by (1),

$$(4) \quad \text{ind}(\sigma; z_j) = \text{ind}(\gamma; z_j) - \text{ind}(\gamma_0; z_j) - c \text{ind}(\partial R_i; z_j) = 0.$$

But σ_{ij} essentially does not appear in σ , that is, σ is equivalent to a cycle τ in which σ_{ij} does not appear; thus $\text{ind}(\sigma; z_k) = \text{ind}(\tau; z_k)$ for all k by Theorem 3.2.5 and the definition of equivalent cycles.

Since z_i and z_j belong to the same component of $\mathbf{C} \setminus \tau^*$, the indices $\text{ind}(\tau; z_i) = \text{ind}(\tau; z_j)$. By (3) and (4), $c = 0$ so that σ_{ij} contributes nothing to $\gamma - \gamma_0$. Exactly the same argument, with z_j replaced by z'_j , shows that if σ'_{ij} is an edge lying between R_i and an unbounded region R_j , then σ'_{ij} contributes nothing to $\gamma - \gamma_0$. But all edges of $\gamma - \gamma_0$ are of the form σ_{ij} or σ'_{ij} , hence $\gamma - \gamma_0$ is equivalent to 0. \square

Theorem 3.3.5. *If f is analytic on the open set $U \subset \mathbf{C}$, then $\int_\gamma f = 0$ for every cycle γ in U such that $\text{ind}(\gamma; z) = 0$ for every z in the complement of U .*

Proof. Construct a partition $a = t_0 < t_1 < \dots < t_n = b$ of the parameter domain of γ and disks $D_1, \dots, D_k \subset U$ such that $\gamma(t) \in D_j$ for $t_{j-1} \leq t \leq t_j$. If γ_j is a polygonal path in D_j from $z_{j-1} = \gamma(t_{j-1})$ to $z_j = \gamma(t_j)$ with edges parallel to the axes, then the integral of f from z_{j-1} to z_j along γ is the same as the integral along γ_j . Thus we may assume without loss of generality that γ is a polygonal cycle with sides parallel to the axes. By the lemma, γ is equivalent to a cycle of the form $\sum n_k \text{ind}(\gamma; z_j) \partial R_j$, where R_j is a rectangle with sides parallel to the axes, and $z_j \in R_j$. If R_j is one of these rectangles and $\overline{R_j}$ is not contained in U , let $z_0 \in \overline{R_j} \setminus U$. Since z_0 is not in γ^* , the segment $[z_0, z_j]$ does not intersect γ^* , hence z_0 and z_j lie in the same component of $\mathbf{C} \setminus \gamma^*$, and so $\text{ind}(\gamma; z_j) = \text{ind}(\gamma; z_0)$. By hypothesis, $\text{ind}(\gamma; z_0) = 0$. Therefore, γ is equivalent to a cycle of the form $\sigma = \sum_{j=1}^l \text{ind}(\gamma; z_j) \partial R_j$, where all $\overline{R_j} \subset U$.

By the equivalence of γ and σ and by Cauchy's Theorem for a Convex Set (Theorem 2.2.6) we obtain that

$$\int_\gamma f = \sum_{j=1}^l \text{ind}(\gamma; z_j) \int_{\partial R_j} f = 0,$$

as advertised. \square

Theorem 3.3.6 (First Cauchy Theorem). *Let $U \subset \mathbf{C}$ be an open subset and let σ be a cycle in U . Then $\int_{\sigma} f = 0$ for every analytic function on U if and only if $\text{ind}(\sigma; z) = 0$ for every $z \notin U$.*

Proof. The “if” part is Theorem 3.3.5. If γ is a cycle in U and $\int_{\gamma} f = 0$ for every analytic function on U , then $\text{ind}(\gamma; z) = 0$ for every z in the complement of U . For if $z_0 \notin U$ and $\text{ind}(\gamma; z_0) \neq 0$, let $f(z) = 1/(z - z_0)$. Then f is analytic on U and, by Theorem 3.2.5, $\int_{\gamma} f = 2\pi i \text{ind}(\gamma; z_0) \neq 0$. \square

Proposition 3.3.7. *Let U be an open subset of \mathbf{C} . The following properties are equivalent.*

- (1) *If σ is a cycle in U , then $\text{ind}(\sigma; z) = 0$ for every point z in the complement of U .*
- (2) *The integral $\int_{\sigma} f = 0$ for every cycle σ in U and every function f analytic on U .*
- (3) *If f is analytic on U , then f has a primitive on U .*
- (4) *If f is analytic on U and f has no zeros on U , then f has an analytic logarithm on U .*

Proof. (1) implies (2). This is First Cauchy Theorem.

(2) implies (3). This is the Existence of Primitive Theorem (Theorem 2.2.4).

(3) implies (4). If f is analytic and has no zeros, then f'/f is analytic on U , and hence has a primitive on U . Theorem 3.1.6 then implies that f has an analytic logarithm on U .

(4) implies (1). If $z_0 \notin U$, then $f(z) = z - z_0$ is analytic and never 0 on U , hence it has an analytic logarithm on U . Now

$$\begin{aligned} \text{ind}(\sigma; z_0) &= \frac{1}{2\pi i} \int_{\sigma} \frac{1}{z - z_0} dz \\ &= 0, \end{aligned}$$

by Theorem 3.1.6, since $f'/f = 1/(z - z_0)$. \square

These properties of an open set U can be described by yet another property which is somewhat more topological.

Definition 3.3.8. An open set $U \subset \mathbf{C}$ is simply connected if both U and \mathbf{P} are connected.

Theorem 3.3.9. *A connected open set U is simply connected if and only if the index $\text{ind}(\sigma; z) = 0$ for every cycle σ in U and every point z in the complement of U .*

Proof. Suppose that $\mathbf{P} \setminus U = A \cup B$, the union of two disjoint open sets. One of these sets, say B , contains the point at infinity, and thus the other is bounded. Let $\text{dist}(A, B) = \delta > 0$. Cover the plane by a net of squares of side $< \delta/\sqrt{2}$. Choose the net so that certain point $a \in A$ lies at the center of one of the squares. Consider the cycle

$$\sigma = \sum_k \partial R_k$$

where the sum is taken over all of the squares of the net which have a point in common with A . It is clear that σ is disjoint from B , by reasons of distance. It is clear that once cancellations are carried out, the cycle σ is equivalent to a cycle which does not intersect A . Finally, since a belongs to exactly one of the squares making up σ , the index $\text{ind}(\sigma; a) = 1$.

Conversely, assume that U is simply connected. Let γ be a closed path in U . Defining $\text{ind}(\gamma; \infty) = 0$, then $\text{ind}(\gamma; \bullet)$ is a continuous function on $\mathbf{P} \setminus \gamma^*$. Since $\gamma^* \subset U$, $\mathbf{P} \setminus U$ is a connected subset of $\mathbf{P} \setminus \gamma^*$; also $\infty \in \mathbf{P} \setminus U$ since $U \subset \mathbf{C}$. Therefore, if $z \notin U$, then z and ∞ belong to the same component of $\mathbf{P} \setminus \gamma^*$, therefore, $\text{ind}(\gamma; z) = \text{ind}(\gamma; \infty) = 0$. \square

The general form of Cauchy's theorem can now be given.

Theorem 3.3.10 (Cauchy's Integral Formula). *Let f be analytic on the open set U . Let σ be a cycle in U such that $\text{ind}(\sigma; z) = 0$ for every z in the complement of U . If $z \in U \setminus \gamma^*$, then*

$$f(z) \text{ind}(\sigma; z) = \frac{1}{2\pi i} \int_{\sigma} \frac{f(w)}{w-z} dw.$$

Proof. Let $g : U \rightarrow \mathbf{C}$ be defined by

$$g(w) = \begin{cases} \frac{f(w) - f(z)}{w-z} & \text{if } w \neq z, \\ f'(z) & \text{otherwise.} \end{cases}$$

Theorem 2.2.11 implies that g is analytic on U and Theorem 3.3.5 implies that $\int_{\sigma} g = 0$. Therefore,

$$\begin{aligned} \frac{1}{2\pi i} \int_{\sigma} \frac{f(w)}{z-w} dz &= f(z) \frac{1}{2\pi i} \int_{\sigma} \frac{1}{w-z} dw \\ &= f(z) \text{ind}(\sigma; z) \end{aligned}$$

by Theorem 3.2.5. □

Chapter 4

Applications

4.1 Singularities

Let f be analytic on $U \setminus \{z_0\}$, where U is an open subset of \mathbf{C} and $z_0 \in U$. In this case z_0 is said to be an isolated singularity of f . The purpose of this section is to determine the behavior of f near z_0 .

Theorem 4.1.1 (Cauchy's Formula for an Annulus). *Let f be analytic on an open set U containing the annulus $r_1 \leq |z - z_0| \leq r_2$, where $0 < r_1 < r_2 < \infty$. Let $\Gamma_i = \{|z - z_0| = r_i, i = 1, 2\}$, oriented counterclockwise. Then for $r_1 < |z - z_0| < r_2$,*

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma_2 - \Gamma_1} \frac{f(w)}{w - z} dw.$$

Proof. By Cauchy's Integral Formula,

$$f(z) \operatorname{ind}(\Gamma_2 - \Gamma_1; z_0) = \frac{1}{2\pi i} \int_{\Gamma_2 - \Gamma_1} \frac{f(w)}{w - z} dw.$$

By the properties of the index, if $r_1 < |z - z_0| < r_2$,

$$\operatorname{ind}(\Gamma_2 - \Gamma_1; z) = \operatorname{ind}(\Gamma_2; z) - \operatorname{ind}(\Gamma_1; z) = 1 - 0 = 1.$$

□

Theorem 4.1.2 (Laurent Series). *Let f be analytic on $U = \{s_1 < |z - z_0| < s_2\}$, where $0 \leq s_1 < s_2 \leq \infty$. Then f is given by*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n, \quad z \in U,$$

where

$$a_n = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(w)}{(w - z_0)^{n+1}} dw$$

and Γ is any circle of radius r , $s_1 < r < s_2$, and center z_0 . The series converges absolutely on U , and converges uniformly on compact subsets.

Proof. Choose $s_1 < r_1 < r_2 < s_2$, and let Γ_1 and Γ_2 be the circles $|w - z_0| = r_1$ and $|w - z_0| = r_2$, respectively. By Cauchy's Integral Formula for an Annulus (Theorem 4.1.1) applied to a point z such that $r_1 < |z - z_0| < r_2$,

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma_2} \frac{f(w)}{w - z} dw - \frac{1}{2\pi i} \int_{\Gamma_1} \frac{f(w)}{w - z} dw.$$

If $w \in \Gamma_2$, then

$$\left| \frac{z - z_0}{w - z_0} \right| = \frac{|z - z_0|}{r_2} < 1,$$

so the series

$$\frac{1}{w - z} = \frac{1}{(w - z_0) \left(1 - \frac{z - z_0}{w - z_0} \right)} = \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(w - z_0)^{n+1}}$$

converges absolutely and uniformly on compact subsets in $|w - z_0| < r_2$, hence uniformly on Γ_2 . By multiplying by the bounded function $\frac{1}{2\pi i} f(w)$ (which preserves uniform convergence) and integrating term by term, we obtain

$$\frac{1}{2\pi i} \int_{\Gamma_2} \frac{f(w)}{w - z} dw = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

where

$$a_n = \frac{1}{2\pi i} \int_{\Gamma_2} \frac{f(w)}{(w - z_0)^{n+1}} dw, \quad n = 0, 1, 2, \dots$$

The second integral is treated similarly. If $w \in \Gamma_1$, then

$$\left| \frac{w - z_0}{z - z_0} \right| \leq \frac{r_1}{|z - z_0|} < 1$$

and the series

$$\frac{-1}{w - z} = \sum_{n=1}^{\infty} \frac{(w - z_0)^{n-1}}{(z - z_0)^n}$$

converges uniformly on Γ_2 . Multiplying by $\frac{1}{2\pi i} f(w)$ and integrating term by term,

$$-\frac{1}{2\pi i} \int_{\Gamma_1} \frac{f(w)}{w - z} dw = \sum_{n=0}^{\infty} \frac{b_n}{(z - z_0)^n}$$

where

$$b_n = \frac{1}{2\pi i} \int_{\Gamma_2} f(w) (w - z_0)^{n-1} dw, \quad n = 1, 2, \dots$$

Replacing the index $n = 1, 2, \dots$ above by $-n = -1, -2, \dots$ and writing

$$a_n = b_{-n} = \frac{1}{2\pi i} \int_{\Gamma_1} f(w) (w - z_0)^{-n-1} dw$$

for $n = -1, -2, \dots$ we obtain the expansion

$$-\frac{1}{2\pi i} \int_{\Gamma_1} \frac{f(w)}{w - z} dw = \sum_{n=-1}^{-\infty} a_n (z - z_0)^n.$$

□

Example 4.1.3. Replacing z by $1/z$ in the power series

$$e^z = 1 + \frac{1}{1!}z + \frac{1}{2!}z^2 + \cdots \quad |z| < \infty$$

we have the Laurent series

$$e^{1/z} = 1 + \frac{1}{1!} \frac{1}{z} + \frac{1}{2!} \frac{1}{z^2} + \cdots \quad 0 < |z| < \infty$$

Note that the series contains no positive powers of z , and that it has an infinite number of negative powers.

Note that the coefficient $a_{-1} = 1$; and according to Laurent's Series theorem, that coefficient is given by

$$a_{-1} = \frac{1}{2\pi i} \int_{\Gamma} e^{1/z} dz$$

where Γ is any positive oriented circle at the origin. Therefore,

$$\int_{\Gamma} e^{1/z} dz = 2\pi i.$$

This method of evaluating certain integrals will be developed further in subsequent sections.

Example 4.1.4. The function $f(z) = 1/(z-i)^2$ is already in the form of a Laurent series where $z_0 = i$. That is

$$f(z) = \sum_{-\infty}^{\infty} a_n (z-i)^n, \quad 0 < |z-i| < \infty,$$

where $a_{-2} = 1$ and all other coefficients are zero. From Laurent series theorem we conclude that if Γ is a positively oriented circle $|z-i| = r$, then

$$\int_{\Gamma} \frac{1}{(z-i)^{n+2}} dz = \begin{cases} 0, & n \neq -2 \\ 2\pi i, & n = -2 \end{cases}$$

Example 4.1.5. The function

$$f(z) = \frac{-1}{(z-1)(z-2)} = \frac{1}{z-1} - \frac{1}{z-2}$$

is analytic on $\mathbf{C} \setminus \{1, 2\}$. It is analytic on the annuli $A_1 = \{|z| < 1\}$, $A_2 = \{1 < |z| < 2\}$ and $A_3 = \{2 < |z| < \infty\}$, and it has a Laurent series on each of them. Their representation can be found with the help of the representation

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad |z| < 1.$$

To find the representation on A_1 , we write

$$f(z) = -\frac{1}{1-z} + \frac{1}{2} \left(\frac{1}{1-(z/2)} \right)$$

and note that $|z| < 1$ and $|z/2| < 1$ on A_1 ,

$$f(z) = -\sum_{n=0}^{\infty} z^n + \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} z^n = \sum_{n=0}^{\infty} (2^{-1-n} - 1) z^n$$

for $|z| < 1$.

As for the representation on A_2 , we write

$$f(z) = \frac{1}{z} \left(\frac{1}{1 - (1/z)} \right) + \frac{1}{2} \left(\frac{1}{1 - (z/2)} \right).$$

Since $|1/z| < 1$ and $|z/2| < 1$ if $1 < |z| < 2$, it follows that

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} + \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} z^n \quad 1 < |z| < 2.$$

By replacing n by $n - 1$ in the first series, and then interchange them, we obtain

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} z^n + \sum_{n=-\infty}^{-1} z^n \quad 1 < |z| < 2.$$

This must be the Laurent series for f on the annulus A_2 because there is at most one such series.

The representation in A_2 is obtained in a similar way. First write

$$f(z) = \frac{1}{z} \left(\frac{1}{1 - (1/z)} \right) + \frac{1}{z} \left(\frac{1}{1 - (2/z)} \right),$$

and note that if $2 < |z| < \infty$, then $|1/z| < 1$ and $|2/z| < 1$. Therefore,

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} - \sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}}, \quad 2 < |z| < \infty$$

That is

$$f(z) = \sum_{n=-\infty}^1 (1 - 2^{n-1}) z^n, \quad 2 < |z| < \infty.$$

Remark. The coefficient a_n of the Laurent expansion of f is not necessarily $\frac{f^{(n)}(z_0)}{n!}$ for $n \geq 0$, because f is only analytic on $s_1 < |z - z_0| < s_2$ and may not have an analytic extension to $|z - z_0| < s_2$.

Theorem 4.1.6. Suppose that $f(z) = \sum_{n=-\infty}^{\infty} b_n (z - z_0)^n$ on $U = \{s_1 < |z - z_0| < s_2\}$, $0 \leq s_1 < s_2 \leq \infty$. Then b_n is given by

$$b_n = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(w)}{(w - z_0)^{n+1}} dw.$$

That is, the Laurent expansion of f on U is unique.

Proof. The power series representing f converges uniformly on compact subsets of U . Multiply both sides by $\frac{1}{(z - z_0)^{k+1}}$ and integrate over Γ to obtain

$$\begin{aligned} \int_{\Gamma} \frac{f(z)}{(z - z_0)^{k+1}} dz &= \sum_{n=-\infty}^{\infty} b_n \int_{\Gamma} (z - z_0)^{n-k-1} dz \\ &= b_k \end{aligned}$$

by the argument in ??.

□

Definition 4.1.7. Let f have an isolated singularity at z_0 , so that f has a Laurent expansion about z_0 of the form

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$$

valid for $0 < |z - z_0| < r$, for some $r > 0$.

The sum of the negative powers of the Laurent series, that is, $\sum_{n=-\infty}^{-1} a_n(z-z_0)^n$ is called the *principal part* of the Laurent expansion of f about z_0 .

If the Laurent expansion contains no negative powers of $(z-z_0)$, then f is said to have a removable singularity at z_0 . In this case, f can be extended to z_0 by setting $f(z_0) = a_0$.

If the principal part has finitely many non-zero terms, that is, if

$$f(z) = \frac{a_{-m}}{(z-z_0)^m} + \cdots + \frac{a_{-1}}{(z-z_0)} + \sum_{n=1}^{\infty} a_n(z-z_0)^n$$

and $a_{-m} \neq 0$, then f is said to have a pole of order m at z_0 (a simple pole if $m = 1$).

In this case $(z-z_0)^m f(z)$ has a removable singularity at z_0 , and $\lim_{z \rightarrow z_0} (z-z_0)^m f(z) = a_{-m} \neq 0$. In this case, setting $f(z_0) = \infty$ we obtain an analytic mapping of U into the Riemann sphere.

Finally, if the principal part of the Laurent series of f about z_0 contains infinitely many non-zero terms, then f is said to have an isolated essential singularity at z_0 .

Lemma 4.1.8. Let f have an isolated singularity at z_0 , and let $M(f, z_0, r) = \max\{|f(z)| \mid |z - z_0| = r\}$. If there are constants $k > 0$ and $\alpha \geq 0$ such that $M(f, z_0, r) \leq kr^{-\alpha}$ for all sufficiently small $r > 0$, then f has either a removable singularity at z_0 or a pole of order $\leq \alpha$.

Proof. The coefficient a_{-n} of the Laurent series can be estimated by using the integral representation

$$|a_{-n}| \leq kr^{n-\alpha}$$

which converges to 0 as $r \rightarrow 0$ if $n > \alpha$; thus $a_{-n} = 0$ in this situation. \square

Theorem 4.1.9 (Casorati-Weierstrass Theorem). Let f have an essential singularity at z_0 . Then for any $r > 0$, the image of the punctured disk $0 < |z - z_0| < r$ is dense in \mathbf{C} .

Proof. Justifying the thesis is equivalent to proving that for any complex number w , the function $g(z) = \frac{1}{f(z) - w}$ is unbounded in any deleted neighborhood of z_0 .

Assume that g is bounded on $V = \{0 < |z - z_0| < b\}$. In particular, $f(z) \neq w$ for all $z \in V$, hence g is analytic on V . Now $M(g, z_0, r) \leq K$, for some constant K and all $0 < r < b$, by assumption. It follows from Lemma 4.1.8 (with $\alpha = 0$) that g has a removable singularity at z_0 . But on V , $f(z) = w + \frac{1}{g(z)}$; therefore, if m is the order of the zero of g at z_0 (setting $m = 0$ if $g(z_0) \neq 0$), then $(z - z_0)^m f(z)$ has a removable singularity at z_0 . Consequently,

$$(z - z_0)^m f(z) = \sum_{n=0}^{\infty} b_n(z - z_0)^n, \quad z \in V$$

and it follows, after dividing by $(z - z_0)^m$, that f has either a removable singularity at z_0 or a pole of order m at z_0 , contradicting the hypothesis. \square

Theorem 4.1.10 (Classification of Singularities). *Let f have an isolated singularity at z_0 .*

- (a) *There is a removable singularity at z_0 if and only if $f(z)$ approaches a finite limit as $z \rightarrow z_0$.*
- (b) *There is a pole of order m at z_0 ($m = 1, 2, \dots$) if and only if $(z - z_0)^m f(z)$ approaches a finite non-zero limit as $z \rightarrow z_0$, and in this case $f(z) \rightarrow \infty$ as $z \rightarrow z_0$.*
- (c) *There is an essential singularity at z_0 if and only if $f(z)$ does not approach a finite or infinite limit as $z \rightarrow z_0$.*

Proof. (a) The “only if” part follows from Definition 4.1.7; the “if” part follows from Theorem 2.2.11.

(b) The “only if” part follows from Definition 4.1.7; for the “if” part, note that if $(z - z_0)^m f(z)$ approaches a finite non-zero limit, then $(z - z_0)^m f(z)$ has a removable singularity at z_0 by (a), hence is given by $\sum_{n=0}^{\infty} b_n (z - z_0)^n$ with $b_0 \neq 0$. Thus, by dividing by $(z - z_0)^m$, it follows that f has a pole of order m at z_0 .

(c) The “only if” part follows from Casorati-Weierstrass Theorem (Theorem 4.1.9); the “if” part follows from (a) and (b). \square

The behavior of a complex function f at ∞ may be studied by considering the function $g(z) = f(1/z)$ at $z = 0$.

4.2 Meromorphic Functions

Definition 4.2.1. The function f has an isolated singularity at ∞ if and only if f is analytic on a set $|z| > r$ and the function $g(z) = f(1/z)$ has an isolated singularity at $z = 0$. Removable singularities, poles, and essential singularities at ∞ are defined similarly.

Definition 4.2.2. A function f on an open subset of the Riemann sphere \mathbf{P} is meromorphic on U if it is analytic on U except for poles and removable singularities.

Example 4.2.3. Let $R(z) = P(z)/Q(z)$ be a rational function, where P and Q are polynomials. The R is a meromorphic function on \mathbf{P} .

Theorem 4.2.4. *If f is meromorphic on \mathbf{P} , then f is a rational function.*

4.3 Calculus of Residues

In this section we present a technique which allows for rapid evaluation of integrals $\int_{\gamma} f$ where γ is a closed path in U and f is analytic on U except for isolated singularities.

Definition 4.3.1. If f has an isolated singularity at z_0 , the coefficient a_{-1} of the Laurent expansion of f about z_0 is called the residue of f at z_0 , and denoted by $\text{res}(f; z_0)$.

Lemma 4.3.2. *Let f be analytic on $U \setminus \{z_0\}$. Let R be a rectangle whose closure is contained in U . If $z_0 \in R$, then*

$$\text{res}(f; z_0) = \frac{1}{2\pi i} \int_{\partial R} f(z) dz.$$

Proof. We may replace ∂R by a circle Γ with center z_0 , by the First Cauchy Theorem (Theorem 3.3.6). Then, Theorem 4.1.2,

$$\begin{aligned} \frac{1}{2\pi i} \int_{\Gamma} f(z) dz &= \sum_{n=-\infty}^{\infty} \frac{a_n}{2\pi i} \int_{\Gamma} (z - z_0)^n dz \\ &= a_{-1} \end{aligned}$$

by the calculation in the proof of the Cauchy Integral Formula for a Circle (Theorem 2.2.7.) \square

Lemma 4.3.3. *Let γ be a closed path in \mathbf{C} , S a subset of \mathbf{C} whose closure is disjoint from γ^* . Assume that whenever w is a limit point of S , then $\text{ind}(\gamma; w) = 0$. Then $\text{ind}(\gamma; z) = 0$ for all but finitely many $z \in S$.*

Proof. The set A consisting of those z satisfying $\text{ind}(\gamma; z) = 0$ is an open subset of $\mathbf{C} \setminus \gamma^*$ which contains $|z| > r$, for r sufficiently large. Therefore $\mathbf{C} \setminus A$ is compact. If infinitely many points of S belong to $\mathbf{C} \setminus A$, then S has a limit point in $\mathbf{C} \setminus A$, contradicting the hypothesis. \square

Theorem 4.3.4 (Residue Theorem). *Let f be analytic on U except for isolated singularities at the points w_1, w_2, \dots . Let γ be a closed path (or cycle) in U such that $\text{ind}(\gamma; z) = 0$ for all z not in U , and such that none of the w_j belong to γ^* . Then*

$$\frac{1}{2\pi i} \int_{\gamma} f = \sum_j \text{ind}(\gamma; w_j) \text{res}(f; w_j).$$

Proof. First note that $\text{ind}(\gamma; w_j) = 0$ for all but finitely many w_j , hence the sum in the statement is finite. Indeed, let $S = \{w_j\}$. If w is a limit point of S , then $w \notin U$ since all the singularities are isolated; thus $\text{ind}(\gamma; w) = 0$. Moreover, the closure of S does not meet γ^* because neither S meets γ^* nor the limit point of S meet γ^* .

Let w_1, \dots, w_n be the singularities for which $\text{ind}(\gamma; w_j) \neq 0$. It may be assumed that γ is a polygonal path with edges parallel to the coordinate axes and contained in $U \setminus \{w_1, w_2, \dots\}$. By a slight modification of γ , if necessary, it may also be assumed that none of the w_1, \dots, w_n lie on the rectangular grid induced by the polygonal path.

By Lemma 3.3.4, the path γ is equivalent to a cycle of the form $\sum_{j=1}^m \text{ind}(\gamma; z_j) \partial R_j$, and for each $k = 1, \dots, n$, the singularity w_k lies in some R_j . The grid may be taken so fine that no two singularities among $\{w_1, \dots, w_n\}$ lie in the same R_j .

Let $V = U \setminus \{w_{n+1}, \dots\}$. If the closure of R_j is not contained in V , let $z_0 \in \overline{R_j} \setminus V$. Since $z_0 \notin V$, it cannot be in $\gamma^* \subset V$, and so the segment $[z_0, z_j]$ does not meet γ^* . It follows that z_0 and z_j lie in the same component of $\mathbf{C} \setminus \gamma^*$, and thus the indexes $\text{ind}(\gamma; z_0) = \text{ind}(\gamma; z_j)$. If $z_0 \notin U$, then $\text{ind}(\gamma; z_0) = 0$ by hypothesis; if $z_0 = w_k$ for some $k > n$, then $\text{ind}(\gamma; z_0) = 0$ by the present construction. In any case, γ is equivalent to a cycle of the form $\sigma = \sum_{j=1}^r \text{ind}(\gamma; z_j) \partial R_j$, where each rectangle R_j has closure contained in V and none of w_1, \dots, w_n is in the boundary of R_j .

Therefore f is defined and continuous on σ^* , and the equivalence of σ and γ yields

$$\int_{\gamma} f = \sum_{j=1}^r \int_{\partial R_j} f.$$

Each of the rectangles appearing in this sum contains at most one of the singularities w_1, \dots, w_n . If $w_k \in R_j$, for some $k = 1, \dots, n$, then $\text{ind}(\gamma; w_k) = \text{ind}(\gamma; z_k)$ and $\int_{\partial R_k} f = 2\pi i \text{res}(f; w_j)$. If R_j contains no w_k , then $\int_{\partial R_j} f = 0$. It follows that

$$\frac{1}{2\pi i} \int f = \sum_{j=1}^n \text{ind}(\gamma; w_j) \text{res}(f; w_j).$$

□

Example 4.3.5. Evaluate the integral $\int_{\gamma} \frac{\text{Log } z}{1 + e^z} dz$, where γ is the curve:

The advantage of expressing an integral in terms of residues is that it is often possible to compute the residues. There is in fact an explicit formula for the residue at a pole.

Theorem 4.3.6 (Residue at a Pole). *If f has a pole of order m at z_0 , then*

$$\text{res}(f; z_0) = \frac{1}{(m-1)!} \lim_{z \rightarrow z_0} \left(\frac{d^{m-1}}{dz^{m-1}} (z - z_0) f(z) \right).$$

In particular, if z_0 is a simple pole,

$$\text{res}(f; z_0) = \lim_{z \rightarrow z_0} (z - z_0) f(z).$$

Proof. Multiply the Laurent expansion

$$f(z) = \frac{a_{-m}}{(z - z_0)^m} + \dots + \frac{a_{-1}}{z - z_0} + \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

by $(z - z_0)^m$, differentiate $m - 1$ times, and take the limit as $z \rightarrow z_0$. □

Lemma 4.3.7. *Let f be analytic at z_0 and have a zero of order m there. Then f'/f has a simple pole at z_0 with $\text{res}(f'/f; z_0) = m$.*

Proof. Write $f(z) = (z - z_0)^m g(z)$, where g is analytic at z_0 and $z_0 \neq 0$. Then

$$\frac{f'(z)}{f(z)} = \frac{m}{z - z_0} + \frac{g'(z)}{g(z)},$$

Since g'/g is analytic at z_0 , the residue

$$\text{res}(f'/f; z_0) = m.$$

□

Example 4.3.8. If f has a pole of order m at z_0 , then f'/f has a simple pole at z_0 with residue $\text{res}(f'/f; z_0) = -m$.

Theorem 4.3.9 (Argument Principle). *Let f be analytic on the connected open set $U \subset \mathbf{C}$, and let γ be a closed path in U such that f is never 0 on γ^* and such that $\text{ind}(\gamma; z) = 0$ for every $z \notin U$. If z_1, \dots are the distinct zeros of f with multiplicities m_1, \dots , then*

$$\text{ind}(f \circ \gamma; 0) = \sum_j m_j \text{ind}(\gamma; z_j).$$

Proof. This follows directly from Theorem 3.2.5. Indeed,

$$\begin{aligned} \operatorname{ind}(f \circ \gamma; 0) &= \frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} dz && \text{by Theorem 3.2.5} \\ &= \sum_j \operatorname{res}(f'/f; z_j) \operatorname{ind}(\gamma; z_j) && \text{by Residue Theorem} \\ &= \sum_j m_j \operatorname{ind}(\gamma; z_j) && \text{by Lemma 4.3.7} \end{aligned}$$

□

Remark. Intuitively, the number of times that $f(z)$ winds about the origin as z traverses the path γ is the number of zeros of f inside γ^* , each zero counted according to its multiplicity and its index with respect to γ .

If U is not assumed to be connected, then we have to add the hypothesis that f is not identically zero on each component of U .

Example 4.3.10. Let $f(z) = (z - a)^m$ and let $\gamma(t) = e^{ikt}$, $0 \leq t \leq 2\pi$ and k an integer. Then f has a zero at a of multiplicity m . The index of a with respect to γ is 0 if $|a| > 1$ and k if $|a| < 1$. Therefore, $\operatorname{ind}(f \circ \gamma; 0) = mk$ if $|a| < 1$ and 0 otherwise.

Theorem 4.3.11 (Generalized Argument Principle). Let f and g be analytic on the open set U , with neither f nor g identically zero on a component of U . Let γ be a closed path on U such that f and g are never 0 on γ^* , and such that $\operatorname{ind}(\gamma; z) = 0$ for all $z \notin U$. If z_1, z_2, \dots are the zeros of f with multiplicities n_1, \dots , and w_1, w_2, \dots are the zeros of g with multiplicities m_1, m_2, \dots , then

$$\operatorname{ind}\left(\frac{f}{g} \circ \gamma; 0\right) = \sum_j n_j \operatorname{ind}(\gamma; z_j) - \sum_k m_k \operatorname{ind}(\gamma; w_k).$$

Proof. Theorem 4.3.9 implies that

$$\operatorname{ind}\left(\frac{f}{g} \circ \gamma; 0\right) = \frac{1}{2\pi i} \int_{\gamma} \frac{(f/g)'}{f/g}.$$

But

$$\frac{(f/g)'}{f/g} = \frac{f'}{f} - \frac{g'}{g}$$

and the result follows from the previous theorem. □

Example 4.3.12. Let $f(z) = \frac{(z-1)(z-3+4i)}{(z+2i)^2}$, and let γ be the circle of center 0 and radius 3. Find $\operatorname{ind}(f \circ \gamma; 0)$.

Solution. The numerator has simple zeros at $z = 1$ and $z = 3 - 4i$. The denominator has a zero of order 2 at $-2i$. The index of 1 and $-2i$ with respect to γ is 1, and the index of $3 - 4i$ is 0. The Generalized Argument Principle yields

$$\begin{aligned} \operatorname{ind}(f \circ \gamma; 0) &= 1 \cdot \operatorname{ind}(\gamma; 1) + 1 \cdot \operatorname{ind}(\gamma; 3 - 4i) - 2 \cdot \operatorname{ind}(\gamma; -2i) \\ &= 1 + 0 - 2 = -1 \end{aligned}$$

Theorem 4.3.13 (Rouche's Theorem). Let f and g be analytic on the connected open set U . Suppose that f has zeros z_1, \dots with multiplicities n_1, \dots and g has zeros w_1, \dots with multiplicities m_1, \dots . Let γ be a path in U such that $\text{ind}(\gamma; z) = 0$ for all $z \notin U$. Assume also that $|f(z) - g(z)| < |f(z)|$ for all z in γ^* . Then $\text{ind}(f \circ \gamma; 0) = \text{ind}(g \circ \gamma; 0)$; hence

$$\sum_j n_j \text{ind}(\gamma; z_j) = \sum_k m_k \text{ind}(\gamma; w_k)$$

Thus f and g have the same number of zeros inside γ^* , counting index and multiplicity.

Proof. The hypothesis that $|f - g| < |f|$ on γ^* implies that f and g are never 0 on γ^* . Therefore, f is never 0 in some neighborhood of γ^* . If $h = 1 + \frac{g-f}{f}$, then $g = hf$ and h is never 0 on γ^* . In fact, $|1 - h| < \delta$ on γ^* , so that the curve $h \circ \gamma$ is contained in the disk $D(1; 1)$.

Now

$$\frac{g'}{g} = \frac{f'}{f} + \frac{h'}{h}$$

so that by Theorem 3.2.5

$$\begin{aligned} \text{ind}(g \circ \gamma; 0) &= \frac{1}{2\pi i} \int_{\gamma} \frac{g'}{g} \\ &= \frac{1}{2\pi i} \int_{\gamma} \frac{f'}{f} + \frac{1}{2\pi i} \int_{\gamma} \frac{h'}{h} \\ &= \text{ind}(f \circ \gamma; 0) + \text{ind}(h \circ \gamma; 0). \end{aligned}$$

But $h \circ \gamma$ is a closed path lying in the disk $D(1, 1)$, and thus $\text{ind}(h \circ \gamma; 0) = 0$ because that disk does not contain 0. □

Example 4.3.14. A polynomial of degree $n \geq 0$ has exactly n zeros, counting multiplicity.

Example 4.3.15. Rouché's theorem gives an easy proof of the following version of the maximum principle: if f is analytic at z_0 , then there is z near z_0 such that $|f(z)| \geq |f(z_0)|$. For if $|f(z)| < |f(z_0)|$ in a disk $|z - z_0| \leq r$, then $f - f(z_0)$ and the constant $f(z_0)$ would have the same number of zeros inside $|z - z_0| = r$. Now $f - f(z_0)$ has a zero at z_0 , and the only way that $f(z_0)$ can have a zero is if $f(z_0) = 0$. This gives $|f(z)| < 0$, a contradiction.

Example 4.3.16. Show that all the zeros of $p(z) = z^4 + 6z + 1$ are inside the circle $|z| = 2$. Moreover, three of the roots are in $1 < |z| < 2$.

Solution. The polynomial $q(z) = z^4 + 6z$ has roots at 0 and at $z = \sqrt[3]{-6}$. For these z , $1 < |z| = \sqrt[3]{6} < 2$. On $|z| = 2$, $|z^4 + 6z + 1| \geq 16 - 13 = 3 > |p(z) - q(z)| = 1$. The zeros of $q(z)$ all have index 1 with respect to $|z| = 2$. The zeros of $p(z)$ have index either 0 or 1, and the sum of their multiplicities is 4.

On $|z| = 1$, $|p(z) - q(z)| = 1$, and $|q(z)| \geq |6z| - 1 = 5$. Therefore q has exactly one root inside $|z| = 1$. To show that there are three roots in $1 < |z| < 2$, look at $|z| = 1 + \varepsilon$ for small ε .

Example 4.3.17. If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is analytic on $|z| \leq R$, and if $a_0 \neq 0$, then f cannot vanish on

$$|z| < \frac{|a_0|R}{M(R) + |a_0|}, \text{ where } M(r) = \sup_{|z|=r} |f(z)|.$$

Theorem 4.3.18. Let $U \subset \mathbf{C}$ be a connected open set whose boundary is a finite collection Γ of simple closed paths. Let f be analytic on $\bar{U} = U \cup \Gamma$, except for a finite number of poles in U , and moreover, f is not zero on Γ . Then

$$\frac{1}{2\pi i} \int_{\Gamma} \frac{f'}{f} = N - P,$$

where N is the number of zeros of f in U and P is the number of poles of f in U , counted according to their multiplicity. Here Γ is the cycle obtained by giving each component γ of Γ the orientation that makes U to be locally on the left of γ .

4.4 Integrals

The Residue Theorem can be used for evaluating integrals.

Rational functions of sin and cos To evaluate

$$\int_0^{2\pi} R(\cos \theta, \sin \theta) d\theta$$

where R is a rational function, we substitute $z = e^{i\theta}$ and note that

$$\cos \theta = \frac{1}{2} \left(z + \frac{1}{z} \right) \quad \text{and} \quad \sin \theta = \frac{1}{2i} \left(z - \frac{1}{z} \right)$$

so that the integral becomes

$$-i \int_{|z|=1} R \left[\frac{1}{2} \left(z + \frac{1}{z} \right), \frac{1}{2i} \left(z - \frac{1}{z} \right) \right] dz.$$

Example 4.4.1. If $a > b > 0$, then show that

$$\int_0^{2\pi} \frac{1}{a + b \cos \theta} d\theta = \frac{2\pi}{\sqrt{a^2 - b^2}}.$$

Solution. The change of variable above yields

$$\int_0^{2\pi} \frac{1}{a + b \cos \theta} d\theta = \frac{2}{ib} \int_{|z|=1} \frac{1}{(z - \alpha)(z - \beta)} dz$$

where

$$\alpha = \frac{-a + \sqrt{a^2 - b^2}}{b} \quad \text{and} \quad \beta = \frac{-a - \sqrt{a^2 - b^2}}{b}.$$

The singularities of $\frac{1}{(z - \alpha)(z - \beta)}$ are simple poles at α and β . Since $\alpha\beta = 1$ and $|\alpha| > |\beta|$, only

the pole α is inside $|z| < 1$. The residue of $\frac{1}{(z - \alpha)(z - \beta)}$ at α is

$$\lim_{z \rightarrow \alpha} (z - \alpha) \frac{1}{(z - \alpha)(z - \beta)} = \frac{1}{\alpha - \beta}.$$

Therefore

$$\int_0^{2\pi} \frac{1}{a + b \cos \theta} d\theta = 2\pi i \frac{2}{bi} \frac{1}{\alpha - \beta}.$$

Improper Integrals Cauchy's theorem can be applied to compute improper integrals, like

$$(1) \quad \int_0^{\infty} \frac{\sin x}{x} dx$$

One defines the improper integrals of a real-valued function $f(x)$ of a real variable $x > 0$ which is bounded near 0 by

$$\int_0^{\infty} f(x) dx = \lim_{R \rightarrow \infty} \int_0^R f(x) dx.$$

In general, f need not be absolutely integrable on $(0, \infty)$, as the example $f(x) = \sin x/x$ above.

To compute (1), consider the function $f(z) = e^{iz}/z$, which is analytic on $z \neq 0$. Let $R > r > 0$ be positive real numbers, and let γ be the closed path depicted below, oriented counterclockwise.

Then, since the index $\text{ind}(\gamma; 0) = 0$, Cauchy's Theorem implies that

$$\int_{\gamma} f(z) = 0$$

Parameterizing γ in the obvious way, this integral can be written as

$$(2) \quad \int_{-R}^{-r} \frac{e^{ix}}{x} dx + \int_r^R \frac{e^{ix}}{x} dx - i \int_0^{\pi} e^{-r \sin \theta + i r \cos \theta} d\theta + i \int_0^{\pi} e^{-R \sin \theta + i R \cos \theta} d\theta.$$

Now

$$\int_{-R}^{-r} \frac{e^{ix}}{x} dx + \int_r^R \frac{e^{ix}}{x} dx = 2i \int_0^R \frac{\sin x}{x} dx$$

and letting $r \rightarrow 0$, the third integral in (2) has the limit $-i \int_0^{\pi} d\theta = -\pi i$. Thus

$$2 \int_0^R \frac{\sin x}{x} dx - \pi + \int_0^{\pi} e^{-R \sin \theta + i R \cos \theta} d\theta = 0.$$

Since this holds true for all $R > 0$, it also holds in the limit as $R \rightarrow \infty$. We now show that the last integral in (2) above approaches 0 as $R \rightarrow \infty$.

First note that

$$|e^{-R \sin \theta + i R \cos \theta}| = e^{-R \sin \theta},$$

so that

$$\left| \int_0^{\pi} e^{-R \sin \theta + i R \cos \theta} d\theta \right| \leq \int_0^{\pi} e^{-R \sin \theta} d\theta = 2 \int_0^{\pi/2} e^{-R \sin \theta} d\theta.$$

If $0 \leq \theta \leq \pi/2$, then $2\theta/\pi \leq \sin \theta$. Thus

$$2 \int_0^{\pi/2} e^{-R \sin \theta} d\theta \leq 2 \int_0^{\pi/2} e^{-2R\theta/\pi} d\theta = \frac{\pi}{R} (1 - e^{-R}).$$

This converges to 0 as $R \rightarrow \infty$, and therefore

$$\lim_{R \rightarrow \infty} 2 \int_0^R \frac{\sin x}{x} dx - \pi = 0,$$

or

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}.$$

Example 4.4.2. Evaluate the integral

$$\int_0^\infty \frac{1}{x^4 + a^4} dx,$$

where $a > 0$.

Solution. Since $1/(x^4 + a^4)$ is even,

$$\int_0^\infty \frac{1}{x^4 + a^4} dx = \frac{1}{2} \int_{-\infty}^\infty \frac{1}{x^4 + a^4} dx.$$

The function $f(z) = \frac{1}{z^4 + a^4}$ has poles at $ae^{k\pi/4}$ ($k = 0, 1, 2, 3$) of which $ae^{\pi/4}$ and $ae^{3\pi/4}$ are in the upper half plane. Integrating along the boundary of a semicircle of radius R and center 0 in the upper half plane, noticing that $|f(x)| \leq 1/R^4$ on $|z| = R$, and letting $R \rightarrow \infty$ we obtain

$$\frac{1}{2} \int_{-\infty}^\infty \frac{1}{x^4 + a^4} dx = \frac{1}{2} 2\pi i \left(\text{res}(f; ae^{\pi/4}) + \text{res}(f; ae^{3\pi/4}) \right) = \frac{\sqrt{2}\pi}{4a^3}.$$

□

4.5 The Open Mapping Theorem

The purpose of this section is to show that a non-constant analytic function maps open sets to open sets, and that a one-one analytic function has an analytic inverse. The next theorem examines the number of solutions of the equation $f(z) = w$, where w is fixed and z ranges over a neighborhood of a zero of f .

Theorem 4.5.1. *Let f be analytic and not identically constant on the disk $D(z_0, r)$, and assume that f has a zero of order k at z_0 . Choose $r_1 < r$ so small that neither f nor f' is zero on $0 < |z - z_0| < r_1$, and let $m = \min\{|f(z)| \mid |z - z_0| = r_1\}$. If $0 < |w| < m$, then the equation $f(z) = w$ has exactly k solutions z in $D(z_0, r_1)$.*

Proof. Note that such r_1 must exist, for otherwise either f or f' has a limit point of zeros, hence f is identically constant on $D(z_0, r)$.

Let $\gamma(t) = z_0 + r_1 e^{it}$, $0 \leq t \leq 2\pi$; Then $|f(z)| \geq m > |w|$ on γ^* . Apply Rouché's Theorem to f and $g = f - w$ to obtain

$$\text{ind}(f \circ \gamma; 0) = \text{ind}((f - w) \circ \gamma; 0).$$

By hypothesis, f has a single 0 inside γ^* (that is, in $|z - z_0| < r_1$) of multiplicity k . By the Argument Principle,

$$\text{ind}((f - w) \circ \gamma; 0) = \sum_j k_j \text{ind}(\gamma; z_j)$$

where the z_j are the zeros of $f - w$ inside γ^* , and k_j is their respective multiplicity.

Because $\text{ind}(\gamma; z_j) = 1$, if there are fewer than k such points, then $k_j > 1$ for at least one index j , and therefore $f - w$ has a zero of order greater than 1 at z_j . This in turn implies that $f'(z_j) = 0$. This leads to a contradiction because the only point inside γ^* where $f' = 0$ is at z_0 , and $f(z_0) - w = w \neq 0$. \square

Theorem 4.5.2. *Let f be analytic and not identically constant on a disk $D(z_0, r)$ and suppose that f has a zero of order k at z_0 . Then there is an open set $U \subset D(z_0, r)$, with $z_0 \in U$, such that for each $z \in U$, $z \neq z_0$,*

(a) $f(z) \neq 0$, and

(b) there are exactly k points z' in $U \setminus \{z_0\}$ such that $f(z') = f(z)$.

Proof. Let r_1 and m be as in previous Theorem 4.5.1. Let $U = D(z_0, r_1) \cap f^{-1}(D(0, m))$. If $z \in U \setminus \{z_0\}$, then $f(z) \neq 0$, by the choice of r_1 , and $|f(z)| < m$ by the choice of U . Theorem 4.5.1 implies that there are exactly k points z' in $D(z_0, r_1)$ with $f(z') = f(z)$. Since $|f(z)| < m$, all such z' belong to $U \setminus \{z_0\}$. \square

Corollary 4.5.3. *Let f be analytic at z_0 . If $f'(z_0) \neq 0$, then there exists a neighborhood of z_0 on which f is one-one. If $f'(z_0) = 0$, then f cannot be one-one in any neighborhood of z_0 .*

Proof. Apply Theorem 4.5.2 to $f(z) - f(z_0)$; note that $f(z) - f(z_0)$ has a zero of order 1 at $z = z_0$ if $f'(z_0) \neq 0$, and a zero of order > 1 at $z = z_0$ if $f'(z_0) = 0$. \square

Theorem 4.5.4 (Open Mapping Theorem). *Let f be analytic on the open set $U \subset \mathbf{C}$, and not identically constant on any component of U . Then $f : U \rightarrow \mathbf{C}$ is an open mapping.*

Proof. It must be proved that if V is an open subset of U , then $f(V)$ is an open subset of \mathbf{C} . Let $z_0 \in V$ and define $g(z) = f(z) - f(z_0)$. Let $D(z_0; r) \subset V$ and construct r_1 and m as in Theorem 4.5.1. (Note that g is not identically constant on $D(z_0; r)$.) It follows from Theorem 4.5.1 that $D(0, m) \subset g(V)$. But then $D(f(z_0), m) \subset f(V)$, for if $|w - f(z_0)| < m$, then there is a point z in V such that $g(z) = w - f(z_0)$, and therefore $f(z) = w$. Thus $f(V)$ is open. \square

Lemma 4.5.5. *Let U and V be open subsets of \mathbf{C} , f a one-one mapping of U onto V , with inverse g . Assume that (1) f is continuous, (2) g is differentiable, and (3) g' is never 0 on U . Then f is differentiable and $f' = \frac{1}{g' \circ f}$.*

Theorem 4.5.6. *Let f be analytic and one-one on an open set U . Then f^{-1} is analytic on the open set $f(U)$.*

Proof. Since f is one-one, it is not identically constant on any component of U , hence $f(U)$ is open by the Open Mapping Theorem. Also by the Open Mapping Theorem, $g = f^{-1}$ is continuous on $V = f(U)$. Because of Theorem 4.5.2, f' is never 0 on U . The result follows from the previous lemma. \square

The next theorem gives a more explicit description of the nature of an analytic function near a zero of order k .

Theorem 4.5.7. *Let f be analytic and non-constant on U . Let $z_0 \in U$ and set $f(z_0) = w_0$. Let k be the order of the zero of $f - w_0$ at z_0 . Then there exists a neighborhood V of z_0 in U , and analytic function φ on V such that*

$$(a) \quad f(z) = w_0 + (\varphi(z))^k \text{ for all } z \text{ in } V.$$

$$(b) \quad \varphi' \text{ has no zero on } V, \text{ and } \varphi \text{ is a one-one mapping of } V \text{ onto } D(0; r).$$

Proof. It may be assumed that U is a convex neighborhood of z_0 , so small that $f(z) \neq w_0$ for all z in $U \setminus \{z_0\}$. Then there is an analytic function g on U such that

$$f(z) - w_0 = (z - z_0)^k g(z)$$

for all z in U and such that g is never zero on U . Therefore g'/g is analytic on U , and since U is convex, it has an analytic logarithm h on U . If $\varphi(z) = (z - z_0) \exp h(z)/k$, then (a) holds for all z in U .

Also $\varphi(z_0) = 0$ and, because this is a simple zero, $\varphi'(z_0) \neq 0$. The rest now follows at once from the Open mapping Theorem. \square

Exercise 4.5.8. Let f be analytic at z_0 , with $f(z_0) = 0$ and $f'(z_0) \neq 0$. Let $\Gamma = \{|z - z_0| = r\}$, where r is chosen small enough so that f is one-one on the disk $D(z_0; s)$ for some $s > r$. If h is analytic on $D(z_0; s)$ and $|w| < \min\{|f(z)| \mid z \in \Gamma\}$, then prove that

$$h(f^{-1}(w)) = \frac{1}{2\pi i} \int_{\Gamma} h(z) \frac{f'(z)}{f(z) - w} dz$$

In particular,

$$f^{-1}(w) = \frac{1}{2\pi i} \int_{\Gamma} z \frac{f'(z)}{f(z) - w} dz$$

Chapter 5

Conformal Mapping

5.1 Preservation of angles

5.2 Mappings of a disk into another

Lemma 5.2.1. *If $|a| \leq 1$ and $f(z) = \frac{z-a}{1-\bar{a}z}$, then f is a one-one analytic map of $D(0;1)$ onto itself. Furthermore, if $|a| < 1$, then f maps the circle $|z| = 1$ one-one onto itself.*

Proof. The function is a linear fractional transformation, so it is one-one. Since $|a| \leq 1$, f is analytic on $D(0;1)$. If $|z| = 1$, then $|z-a| = |z||1-\bar{a}z|$ because $\bar{z} = 1/z$, hence $|f(z)| = 1$. The inverse of f is given by $g(w) = \frac{w+a}{1+\bar{a}w}$. Thus g is of the same form as f , and so it is one-one and $|g(w)| = 1$ if $|w| = 1$. It follows that f maps $|z| = 1$ one-one onto itself.

By the Maximum principle, f maps $D(0;1)$ into itself, and so does g . But if $f(D(0;1)) \subset D(0;1)$ and $g(D(0;1)) \subset D(0;1)$, then $f(D(0;1)) = D(0;1)$. \square

Exercise 5.2.2. If $f(z) = \frac{R(z-z_0)}{R^2-\bar{z}_0z}$, where $|z_0| < R$, then f is a one-one analytic mapping of the disk $D(0;R)$ onto the disk $D(0;1)$; also, f maps the circle $|z| = R$ one-one onto $|z| = 1$.

Theorem 5.2.3 (Schwarz's lemma). *Let f be analytic map on $D(0,1)$ satisfying $|f(z)| \leq 1$ and $f(0) = 0$. Then $|f(z)| \leq |z|$ for all z in $D(0;1)$, and $|f'(0)| \leq 1$. Furthermore, if $|f(z_0)| = |z_0|$ for some $z_0 \neq 0$, or if $|f'(0)| = 1$, then f is of the form $f(z) = az$ for some complex number a with $|a| = 1$.*

Proof. Let $g(z) = \frac{f(z)}{z}$, $z \neq 0$; $g(0) = f'(0)$. Then g is analytic on $D(0;1)$. If $r < 1$, then $|g(z)| \leq 1/r$ for $|z| = r$, so also $|g(z)| \leq 1/r$ for $|z| \leq r$, by the Maximum Principle. Letting $r \rightarrow 1$, it follows that $|g(z)| \leq 1$ if $|z| < 1$, hence that $|f(z)| \leq |z|$.

If $|g(z_0)| = 1$ for some $z_0 \neq 0$, then g must be constant by the Maximum Principle. If this happens, $f(z) = az$ for some a with $|a| = 1$.

In view of the power series expansion

$$g(z) = \frac{f(z)}{z} = f'(0) + \frac{g'(0)}{1!}z + \dots$$

it follows that $|f'(0)| = |g(0)| \leq 1$. Equality $|f'(0)| = 1$ implies that $|g(0)| = 1$, hence that $f(z) = az$ as before. \square

Corollary 5.2.4. Let f be an analytic mapping of $D(0;R)$ into $D(0;M)$, with $f(z_0) = w_0$. Then

$$\left| \frac{M(f(z) - w_0)}{M^2 - \overline{w_0}f(z)} \right| \leq \left| \frac{R(z - z_0)}{R^2 - \overline{z_0}z} \right|$$

for all z in $D(0;R)$.

Proof. Let $T(z)$ and $S(w)$ be the mappings defined by

$$T(z) = \frac{R(z - z_0)}{R^2 - \overline{z_0}z}, \quad \text{and} \quad S(w) = \frac{M(w - w_0)}{M^2 - \overline{w_0}w}$$

Then $g = S \circ f \circ T^{-1}$ is an analytic mapping of $D(0;1)$ into $D(0;1)$ with $|F(0)| = S(f(T^{-1}(0))) = 0$, because $T^{-1}(0) = z_0$. By Schwarz lemma, $|F(z)| \leq |z|$ for $|z| < 1$, and hence $|S(f(z))| \leq |T(z)|$ for $|z| < R$. \square

Remark. If equality holds at one point other than z_0 , then $S \circ f \circ T^{-1}(z) = az$ for some a with $|a| = 1$, and so $f(z) = S^{-1}(aT(z))$.

Definition 5.2.5. For a with $|a| < 1$, let

$$\phi_a = \frac{z - a}{1 - \overline{a}z}.$$

Theorem 5.2.6. Fix a in $D(0;1)$. Then ϕ_a is a one-one analytic mapping of $D(0;1)$ onto $D(0;1)$ which carries $|z| = 1$ one-one onto itself. Furthermore, the inverse of ϕ_a is ϕ_{-a} , and

$$\phi'_a(0) = 1 - |a|^2, \quad \phi'_a(a) = \frac{1}{1 - |a|^2}$$

Proof. The mapping ϕ_a is analytic on the whole plane, except for a pole at $1/\overline{a}$, which lies outside $D(0;1)$. Straightforward substitution shows that

$$\phi_{-a} \circ \phi_a(z) = z.$$

\square

An extremal Problem Suppose that a and b are complex numbers satisfying $|a| < 1$ and $|b| < 1$. How large can $|f'(a)|$ be subject to the conditions that f is analytic on $|z| < 1$, $|f(z)| \leq 1$ and $f(a) = b$?

To solve this problem, put

$$g(z) = \phi_b \circ f \circ \phi_{-a}.$$

The g is analytic on $|z| < 1$, it is bounded as $|g(z)| \leq 1$, and $g(0) = 0$. Apply Schwarz lemma to get $|g'(0)| \leq 1$. By the Chain rule,

$$g'(0) = \phi'_b(b)f'(a)\phi'_{-a}(0)$$

and therefore

$$|f'(a)| \leq \frac{1 - |b|^2}{1 - |a|^2}.$$

Equality can occur if and only if $|g(0)| = 1$, in which case $g(z) = \lambda z$ for some λ with $|\lambda| = 1$, and

$$f(z) = \phi_{-b}(\lambda\phi_a(z)).$$

Remark. Note that no smoothness condition was imposed on the behavior of f near $|z| = 1$ (such as continuity on $|z| = 1$). Nevertheless, the functions which maximize $|f'(a)|$ are actually rational functions. Note also that the extremal functions map $D(0; 1)$ onto (not just into) $D(0; 1)$ and that they are one-one. This would motivate the proof of the Riemann mapping Theorem.

At this moment we show how this extremal problem can be used to characterize the one-one maps of a disk onto itself.

Theorem 5.2.7. *Suppose that f is an analytic one-one mapping of the unit disk $D(0; 1)$ onto itself. Let a be such that $f(a) = 0$. Then there is a constant λ , $|\lambda| = 1$, such that*

$$f(z) = \lambda \phi_a(z)$$

for all z in $D(0; 1)$

Proof. Let g be the inverse of f , defined by $g(f(z)) = z$. Since f is one-one, f' has no zero on U , and so g is analytic on U . By the Chain Rule,

$$g'(0)f'(a) = 1$$

The solution to the extremal problem applied to f and to g yields

$$|f'(a)| \leq \frac{1}{1 - |a|^2}, \quad |g'(0)| \leq 1 - |a|^2.$$

Therefore equality must hold. Now apply the solution to the extremal problem with $b = 0$. □

5.3 Normal Families

Definition 5.3.1. Let U be an open subset of \mathbf{C} . Denote by $\mathcal{A}(U)$ the space of all analytic functions on U , and by $\mathcal{C}(U)$ the space of all continuous functions from U to \mathbf{C} .

We will study metric properties of these spaces. They have a topology, uniform convergence on compact subsets of U . That is, $f_n \rightarrow f$ if for every compact set $K \subset U$, the sequence of numbers

$$\|f_n - f\|_K = \sup_{z \in K} |f_n(z) - f(z)| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

In fact, this topology is that of a metric space. To define the corresponding distance, let

$$K_n = \{z \in \mathbf{C} \mid |z| \leq n \text{ and } |z - w| \geq 1/n \text{ for all } w \in \mathbf{C} \setminus U\}.$$

Then the K_n form an expanding sequence ($K_n \subset K_{n+1}$) of compact subset of U , and $\bigcup_n K_n = U$. Furthermore, if K is a compact subset of U , then K is bounded and is at positive distance from the complement of U ; hence $K \subset K_n$ for sufficiently large n .

If $f, g \in \mathcal{C}(U)$, define

$$d(f, g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|f - g\|_{K_n}}{1 + \|f - g\|_{K_n}}.$$

This is a metric because

$$\frac{x}{1 + x + y} \leq \frac{x}{1 + x}$$

for all $x, y \geq 0$, so

$$\frac{x + y}{1 + x + y} \leq \frac{x}{1 + x} + \frac{y}{1 + y}.$$

Theorem 5.3.2. *Convergence in $\mathcal{C}(U)$ with respect to the metric d is the same as uniform convergence on compact sets.*

Proof. Suppose that $d(f_n, f) \rightarrow 0$. Then $\|f_j - f\|_{K_n} \rightarrow 0$ as $j \rightarrow \infty$ for each n . If K is a compact subset of U , then K is contained in some K_n . Hence $\|f_j - f\|_K \leq \|f_j - f\|_{K_n} \rightarrow 0$ as $j \rightarrow \infty$ also.

Conversely, suppose $f_j \rightarrow f$ uniformly on compact subsets of U . Given $\varepsilon > 0$, choose N such that $\sum_{n>N} \frac{1}{2^n} < \varepsilon$. Then choose J so that $j \geq J$ implies

$$\frac{\|f_j - f\|_{K_n}}{1 + \|f_j - f\|_{K_n}} < \frac{\varepsilon}{2} \quad \text{for } n = 1, 2, \dots, N.$$

(Note that if this inequality hold for $n = N$ it also holds for $n \geq N$ because $K_n \subset K_m$ if $n < m$.) It follows that $d(f_j, f) < \varepsilon$ if $j \geq J$. \square

Remark. The same argument shows that $\{f_n\}$ is a Cauchy sequence in $\mathcal{C}(U)$, that is, $d(f_n, f_m) \rightarrow 0$ as $n, m \rightarrow \infty$, if and only if $f_n - f_m$ converges to 0 uniformly on compact subsets of U .

Example 5.3.3. Look at $f_n(z) = z^n$ and $f_n(z) = nz$.

Theorem 5.3.4. *With respect to the metric d the space $\mathcal{C}(U)$ is complete, and $\mathcal{A}(U)$ is a closed subspace of $\mathcal{C}(U)$.*

Proof. If $d(f_n, f_m) \rightarrow 0$, then for each $z \in U$, $\{f_n(z)\}$ is a Cauchy sequence of complex numbers (the set $K = \{z\}$ is compact). Define $f(z) = \lim_n f_n(z)$.

If $K \subset U$ is compact and $\varepsilon > 0$, then $|f_n(z) - f_m(z)| \leq \varepsilon$ for all z in K , provided $n, m \geq N$. Fix $n \geq N$ and let $m \rightarrow \infty$ to obtain $|f_n(z) - f(z)| \leq \varepsilon$ for all z in K and all $n \geq N$. This says that $f_n \rightarrow f$ uniformly on K .

Choosing K to be a closed disk about a point in U gives continuity of f , and thus completeness of $\mathcal{C}(U)$.

Theorem ?? showed that if f_1, f_2, \dots , are analytic functions on U and $f_n \rightarrow f$ uniformly on compact subsets of U , then f is analytic on U , that is, $\mathcal{A}(U)$ is closed in $\mathcal{C}(U)$. \square

Theorem 5.3.5 (Hurwitz's Theorem). *Let $\{f_n\}$ be a sequence in $\mathcal{A}(U)$ such that $f_n \rightarrow f$ uniformly on compact subsets of U . Suppose that the closure of the disk $D(z_0; r)$ is contained in U and that f is not zero on $|z - z_0| = r$. Then there is a positive integer N such that, for all $n \geq N$, f_n and f have the same number of zeros in $D(z_0; r)$.*

Proof. As remarked, f is analytic on U . Let $\varepsilon = \min\{|f(z)| \mid |z - z_0| = r\}$. Choose N sufficiently large so that if $n \geq N$, then $|f_n(z) - f(z)| < \varepsilon \leq |f(z)|$ for $|z - z_0| = r$. The result follows from Rouché's Theorem. \square

Corollary 5.3.6. *Let U be connected and $f_n \rightarrow f$ uniformly on compact subsets of U . If the f_n are never 0 on U , then f is either never 0 or identically 0 on U .*

Proof. If $f(z_0) = 0$ and f is not identically 0, there is a closed disk $|z - z_0| \leq r$ such that f is not 0 on $|z - z_0| = r$. It follows from Hurwitz's Theorem that, for sufficiently large n , the function f_n must have a zero in $D(z_0; r)$. \square

Corollary 5.3.7. *Let U be connected and $f_n \rightarrow f$ uniformly on compact subsets of U . If the f_n are one-one on U , then f is either one-one or identically constant on U .*

Proof. The fact that a function f is one-one on U may be rephrased by saying that for each z_0 in U , the function $f - f(z_0)$ is never 0 on $U \setminus \{z_0\}$.

Choose a point z_0 in U and set $g_n = f_n - f_n(z_0)$. Then g_n is analytic on U and converges uniformly on compact subsets of $U \setminus \{z_0\}$ to $f - f(z_0)$. Each function g_n is never zero on $U \setminus \{z_0\}$ because each f_n is one-one. The set $U \setminus \{z_0\}$ is connected, so that the Corollary above implies that $f - f(z_0)$ is either identically constant or never 0 on U . If $f - f(z_0)$ is identically constant on $U \setminus \{z_0\}$ for a single z_0 , then f is constant on U . Since z_0 was arbitrary the result follows. \square

The next task is to identify the compact subsets of $\mathcal{A}(U)$. This requires the introduction of a notion of boundedness for subsets $F \subset \mathcal{A}(U)$, and to show that if F is bounded, any sequence of functions in F has a subsequence which converges uniformly on compact subsets of U . This will lead to the result that a subset of $\mathcal{A}(U)$ is bounded if and only if it is compact.

Definition 5.3.8. A set $\mathcal{F} \subset \mathcal{C}(U)$ is bounded if there is $f \in \mathcal{C}(U)$ and $R > 0$ such that $d(f, f_0) \leq R$ for all $f \in \mathcal{F}$.

That is, a set $\mathcal{F} \subset \mathcal{C}(U)$ is bounded if for each compact subset $K \subset U$,

$$\sup\{\|f\|_K \mid f \in \mathcal{F}\} < \infty.$$

This means that the functions in \mathcal{F} are uniformly bounded on each compact subset of U .

Definition 5.3.9. A subset $\mathcal{F} \subset \mathcal{C}(U)$ is equicontinuous at $z_0 \in U$ if for every $\varepsilon > 0$ there is $\delta > 0$ such that for all $z \in U$ and all $f \in \mathcal{F}$, if $|z - z_0| < \delta$, then $|f(z) - f(z_0)| < \varepsilon$.

Theorem 5.3.10 (Bounded implies equicontinuous). Let \mathcal{F} be a bounded subset of $\mathcal{A}(U)$. Then \mathcal{F} is equicontinuous at each point in U .

Proof. Let $r > 0$ be such that the closed disk $|z - z_0| \leq r$ is contained in U . If $z, z' \in D(z_0; r/2)$ and $f \in \mathcal{F}$, then Cauchy's Integral Formula for a Circle applied to $\Gamma = \{|z - z_0| = r\}$ implies that

$$\begin{aligned} f(z) - f(z') &= \frac{1}{2\pi i} \int_{\Gamma} f(w) \left[\frac{1}{w - z} - \frac{1}{w - z'} \right] dw \\ &= \frac{z - z'}{2\pi i} \int_{\Gamma} \frac{f(w)}{(w - z)(w - z')} dw. \end{aligned}$$

If $M = \sup\{\|f\|_{\Gamma} \mid f \in \mathcal{F}\}$, then $M < \infty$ by the hypothesis and so

$$|f(z) - f(z')| \leq \frac{1}{2\pi} |z - z'| M \frac{1}{(r/2)^2} 2\pi r$$

from what the proof follows. \square

Theorem 5.3.11. Let $\mathcal{F} \subset \mathcal{C}(U)$ be equicontinuous on U . If f_n is a sequence in \mathcal{F} such that $f_n \rightarrow f$ pointwise in U , then f is continuous on U , and in fact $f_n \rightarrow f$ uniformly on compact subsets of U .

Moreover, if $f_n(z) \rightarrow f(z)$ only for the points z in a dense subset of U , then in fact $f_n(z)$ converges to a limit $f(z)$ for all z in U , and, by the above, the limit f is continuous in U and $f_n \rightarrow f$ uniformly on compact subsets of U .

Proof. Let K be a compact subset of U , and let $\varepsilon > 0$. If $z \in K$, then there is a disk $D(z) \subset U$ such that if $z' \in D(z)$, then $|g(z) - g(z')| \leq \varepsilon/3$ for all g in \mathcal{F} . In particular, $|f_n(z) - f_n(z')| \leq \varepsilon/3$ for $n = 1, 2, \dots$, and so $|f(z) - f(z')| \leq \varepsilon/3$, implying continuity of f at z , and thus in U .

Because K is compact, there are finitely many points z_1, \dots, z_m in K such that $K \subset \bigcup_{j=1}^m D(z_j)$. For each $j = 1, \dots, m$,

$$|f(z) - f_n(z)| \leq |f(z) - f(z_j)| + |f(z_j) - f_n(z_j)| + |f_n(z_j) - f_n(z)|.$$

If $z \in K$, then z is in one of the disks $D(z_j)$ and so the first and third terms in the sum (*) are each at most $\varepsilon/3$, for all n . The second is less than or equal to $\varepsilon/3$ if n is sufficiently large, say $n \geq N_j$, by pointwise convergence. It follows that if $n \geq \max\{N_1, \dots, N_m\}$, then $|f(z) - f_n(z)| \leq \varepsilon$ for all $z \in K$. \square

Theorem 5.3.12 (Montel's Theorem). *Let \mathcal{F} be a bounded subset of $\mathcal{A}(U)$. If $\{f_n\}$ is a sequence in \mathcal{F} , then there is a subsequence $\{f_{n_k}\}$ which converges uniformly on compact subsets of U (to a limit function f which is analytic on U).*

Proof. Let $S = \{z_1, z_2, \dots\}$ be a countable and dense subset of U (for example, all points in U whose real and imaginary parts are rational numbers). Because \mathcal{F} is bounded, there is a constant M_1 such that $|f(z_1)| \leq M_1$ for all $f \in \mathcal{F}$. Therefore there is a subsequence f_{11}, f_{12}, \dots of $\{f_n\}$ such that the sequence of numbers $\{f_{1n}(z_1)\}$ converges to a limit w_1 . Repeat the process with the point z_2 and the sequence $\{f_{1n}\}$ to obtain a subsequence f_{21}, f_{22}, \dots of $\{f_{1n}\}$ such that $\{f_{2n}(z_2)\}$ converges to a limit w_2 . Continue in this fashion to obtain a subsequence $f_{k,1}, f_{k,2}, \dots$ of the sequence $\{f_{k-1,n}\}$ such that $f_{k,n}(z_k)$ approaches a limit w_k as $n \rightarrow \infty$.

Let $g_n(z) = f_{nn}(z)$, $n = 1, 2, \dots$. Then for each k , g_k, g_{k+1}, \dots is a subsequence of $f_{k,1}, f_{k,2}, \dots$ and thus $g_n(z_k) \rightarrow w_k$ as $n \rightarrow \infty$, for all $k = 1, 2, \dots$. Previous theorem implies that there is a function f which is analytic on U and such that $g_n \rightarrow f$ uniformly on compact subsets of U . \square

Theorem 5.3.13 (Compactness Criterion). *Let $\mathcal{F} \subset \mathcal{A}(U)$. Then \mathcal{F} is compact if and only if \mathcal{F} is closed and bounded.*

Proof. Compactness of a subset of a metric space is equivalent to the fact that every sequence in the subset has a subsequence which converges to a point in the subset.

If $\mathcal{F} \subset \mathcal{A}(U)$ is closed and bounded, then Morera's Theorem shows that every sequence in \mathcal{F} has a subsequence which converges to an analytic function $f \in \mathcal{A}(U)$. Since \mathcal{F} is closed, $f \in \mathcal{F}$.

Conversely, suppose that \mathcal{F} is compact. Any compact subset of a metric space is closed. To show boundedness, let $K \subset U$ be compact. Then the map $f \in \mathcal{C}(U) \mapsto \|f\|_K$ is continuous, so it takes compact sets to compact subsets of the reals. \square

Example 5.3.14. Let U be an open subset of the plane and let \mathcal{F} be the collection of analytic functions f on U which have non-negative real part. Then \mathcal{F} is compact. Use the mapping $z \mapsto \frac{z-i}{z+i}$ of the upper half plane into the unit disc.

Compactness considerations typically arise in extremal problems. It will be more appropriate to formalize this as follows.

Definition 5.3.15. Let \mathcal{F} be a subset of $\mathcal{C}(U)$. A continuous mapping $J: \mathcal{F} \rightarrow \mathbf{C}$ is called a functional on \mathcal{F} .

Thus $J : \mathcal{F} \rightarrow \mathbf{C}$ is a functional if $J(f_n) \rightarrow J(f)$ for every sequence f_n in \mathcal{F} which converges uniformly on compact sets to $f \in \mathcal{F}$.

Example 5.3.16. Let $U \subset \mathbf{C}$ be an open set and let $z_0 \in U$. The mapping $J(f)$ which associates to an analytic function f on U the derivative $f'(z_0)$ is a functional on $\mathcal{A}(U)$. More generally, the mapping $f \mapsto f^{(n)}(z_0)$ is a functional on $\mathcal{A}(U)$.

Theorem 5.3.17. Let \mathcal{F} be a (non-empty) compact subset of $\mathcal{C}(U)$ and let J be a functional on \mathcal{F} . Then there exists $f \in \mathcal{F}$ such that

$$|J(f)| \geq |J(g)|$$

for all $g \in \mathcal{F}$. (Non-empty is needed because $\inf \emptyset = \infty$.)

Proof. Since \mathcal{F} is compact and J is continuous, the image $J(\mathcal{F})$ is a compact subset of \mathbf{C} . Thus $M = \sup_{f \in \mathcal{F}} |J(f)| < \infty$ and, by the definition of supremum, there is a sequence f_n in \mathcal{F} such that $|J(f_n)| \rightarrow M$. Since \mathcal{F} is compact, this sequence has a subsequence which converges uniformly on compact subsets of U to some $f \in \mathcal{F}$. Then $|J(f)| = \lim_n |J(f_n)| = M \geq |J(g)|$ for all $g \in \mathcal{F}$. \square

Theorem 5.3.18. Let \mathcal{F} be a non-empty compact set of analytic functions on U . If $z_0 \in U$, then there exists g in \mathcal{F} such that $|g'(z_0)| \geq |f'(z_0)|$ for all $f \in \mathcal{F}$.

Proof. The mapping $f \mapsto |f'(z_0)|$ of \mathcal{F} into the reals is a functional on \mathcal{F} . \square

Theorem 5.3.19. Let $U \subset \mathbf{C}$ be a connected open subset. Let $z_0 \in U$, let $b > 0$ be a fixed real number, and let \mathcal{F} be the collection of analytic one-one functions f on U such that $\sup_{z \in U} |f(z)| \leq 1$ and $|f'(z_0)| \geq b$. Then \mathcal{F} is compact.

Proof. It is clear that \mathcal{F} is bounded. We must show that \mathcal{F} is closed, that is, if $f_n \in \mathcal{F}$ and $f_n \rightarrow f$ uniformly on compact subsets, then $f \in \mathcal{F}$. It is immediate that $\sup_{z \in U} |f(z)| \leq 1$. Because the f_n are one-one on the connected set U , the Corollary to Hurwitz's theorem implies that f is either one-one or identically constant on U . Since $|f'_n(z_0)| \rightarrow |f'(z_0)|$, then $|f'(z_0)| \geq b > 0$, excluding the possibility that f be constant. \square

5.4 The Riemann Mapping Theorem

In this section U will denote an open, connected, and proper subset of \mathbf{C} such that every function f which is analytic and never 0 on U has an analytic square root on U . This is true if U is simply connected in the sense that every analytic f which is never 0 on U has an analytic logarithm on U .

The idea of the proof is as follows. The family \mathcal{F} of all functions which are one-one and analytic on U and $|f| \leq 1$ is compact. Fix a point $z_0 \in U$ and try to find a function in \mathcal{F} which maximizes the dilation $|f'(z_0)|$ at z_0 . This extremal problem can be solved because the family \mathcal{F} is compact and $J(f) = |f'(z_0)|$ is a continuous functional on \mathcal{F} . The solution to this problem is then shown to map the open set U onto the unit disk.

Lemma 5.4.1. Let U be an open connected proper subset of \mathbf{C} such that every analytic function on U has an analytic square root. Then there is a one-one analytic mapping of U into the unit disk $D(0; 1)$.

Proof. Fix a point $a \notin U$. By hypothesis, there is a function h analytic on U such that $h^2(z) = z - a$, necessarily h is one-one because $z - a$ is one-one. By the Open Mapping Theorem, $h(U)$ contains an open disk $D(z_0; r)$. Note also that $0 \notin D(z_0; r)$, for if so, then $h(z) = 0$ for some $z \in U$, hence $z = a \in U$, a contradiction. It follows that $h(U) \cap D(-z_0; r) = \emptyset$, for if $h(z) = w \in D(-z_0; r)$ for some $z \in U$, then we can find $z' \in U$ such that $h(z') = -w$. But then $h^2(z) = h^2(z')$, which implies that $z = z'$ and that $w = 0$, a contradiction.

Therefore, $|h(z) + z_0| \geq r$ for all $z \in U$. Define $f(z) = \frac{kr}{h(z) + z_0}$, where $0 < |k| < 1$. \square

Lemma 5.4.2. *Let \mathcal{F} be the collection of one-one analytic mappings of U into $D(0; 1)$. Let $g \in \mathcal{F}$, and let z_0 be a fixed point in U . If g is not onto, then there is a function $f \in \mathcal{F}$ such that $|f'(z_0)| > |g'(z_0)|$.*

Proof. If a is not in $g(U)$ and $|a| < 1$, then $\phi_a \circ g$ is never 0 on U . Thus there is $h \in \mathcal{A}(U)$ such that $h^2 = \phi_a \circ g$. Since ϕ_a and g are one-one, so is h ; since $h^2(U) \subset D(0; 1)$, we have $h(U) \subset D(0; 1)$. That is $h \in \mathcal{F}$.

Let $f = \phi_b \circ h$, where $b = h(z_0)$. Then $f \in \mathcal{F}$ and

$$g = \phi_{-a} \circ h^2 = \phi_{-a} \circ (\phi_{-b} \circ f)^2 = \phi_{-a} \circ S \circ \phi_{-b} \circ f$$

where $S(z) = z^2$. Let $T = \phi_{-a} \circ S \circ \phi_{-b}$. Then by the Chain rule

$$g'(z_0) = T'(f(z_0))f'(z_0) = T'(\phi_b(h(z_0)))f'(z_0) = T'(0)f'(z_0).$$

Now T is an analytic mapping of $D(0; 1)$ into itself which is not one-one (because the square root S is not one-one). By the extremal problem considered earlier applied to the mapping T and the points 0 and $T(0)$, $|T'(0)| \leq 1 - |T(0)|^2 \leq 1$, with equality if and only if T is one-one. Therefore

$$|g'(z_0)| = |T'(0)||f'(z_0)| < |f'(z_0)|.$$

\square

Theorem 5.4.3 (Riemann's Mapping Theorem). *With U as above, there is a one-one analytic mapping of U onto the unit disk $D(0; 1)$.*

Proof. Let h be a one-one analytic mapping of U into $D(0; 1)$. Pick $z_0 \in U$ and let $b = |h'(z_0)| > 0$. Let \mathcal{F} be the collection of one-one analytic mapping of U into $D(0; 1)$ such that $|f'(z_0)| \geq b$. Last lemma implies that \mathcal{F} is not empty, and Corollary to Montel's Theorem (which requires connectedness) implies that \mathcal{F} is compact. Thus there is a function $f \in \mathcal{F}$ such that $|f'(z_0)| \geq |g'(z_0)|$ for all $g \in \mathcal{F}$. If f is not onto, then because the Lemma above there exists g in \mathcal{F} such that $|g'(z_0)| > |f'(z_0)| \geq b$, a contradiction. \square

Theorem 5.4.4 (Simply Connected Sets). *Let U be a connected open subset of \mathbf{C} . Then the following are equivalent.*

- Every analytic f which is never 0 on U has an analytic square root.
- U is homeomorphic to $D(0; 1)$.
- Every analytic non-zero f has an analytic logarithm on U .

Proof. (1) implies (2) If $U \neq \mathbf{C}$ this is the Riemann mapping Theorem. If $U = \mathbf{C}$, define $h(z) = z/(1 + |z|)$.

(2) implies (3) If f is analytic and nowhere 0 on $D(0; 1)$ or on \mathbf{C} , then f'/f has a primitive on $D(0; 1)$ (or on \mathbf{C}) because the unit disc (or the plane) is convex. Thus f has an analytic logarithm.

For $U \neq \mathbf{C}$, pick an analytic one-one map h of U onto $D(0; 1)$. If f is analytic and never 0 on U , then $g = f \circ h^{-1}$ is analytic and never 0 on $D(0; 1)$, so there is an analytic function G on $D(0; 1)$ such that $e^G = g$. Then $G \circ h$ is analytic in U and $e^{G \circ h(z)} = g(h(z)) = f(z)$ for all $z \in U$.

(3) implies (1) Done. □

5.5 The Schwarz-Christoffel Formula

5.6 Complex Dynamics