

MATH 650. PRACTICE MIDTERM

1. Let X be a set.
 - (a) Carefully define the concepts σ -ring and σ -field of subsets of X .
 - (b) Let \mathcal{R} be the collection of countable subsets of X . Show that \mathcal{R} is a σ -ring, and determine when \mathcal{R} is a σ -field.
2. Let X be a countable infinite set, and let \mathcal{F} be the collection of its finite subsets and their complements. For $A \in \mathcal{F}$, let $\mu(A) = 0$ if A is finite and $\mu(A) = 1$ if the complement of A is finite.
 - (a) Show that \mathcal{F} is a field.
 - (b) Show that μ is finitely additive. Is it countably additive?
3. Let (X, \mathcal{F}, μ) be a probability space (with \mathcal{F} a σ -field) and let B_1, B_2, \dots , be a countable collection of sets from \mathcal{F} such that $\sum_{n=1}^{\infty} \mu(B_n) < \infty$. Show that $\mu(\limsup B_n) = 0$.
4. Let X be a set and \mathcal{F} a σ -field of subsets of X .
 - (a) Define the concept of *measurable function* $f : X \rightarrow [-\infty, +\infty]$.
 - (b) For $X = \mathbf{R}$ and \mathcal{F} the Lebesgue measurable sets, show that the function $f : \mathbf{R} \rightarrow [-\infty, +\infty]$ given by

$$f(x) \begin{cases} -x & \text{if } x < 0, \text{ and} \\ +\infty & \text{if } x \geq 0, \end{cases}$$

5. Let (X, \mathcal{F}, μ) be a probability space (with \mathcal{F} a σ -field).
 - (a) Define the concepts: Two sets A and B from \mathcal{F} are independent; a countable collection $A_n, n = 1, 2, \dots$, of sets from \mathcal{F} is independent.
 - (b) Let X be the unit interval, \mathcal{F} the Lebesgue measurable sets and μ the Lebesgue measure. Let B_n be the subset of the unit interval corresponding to the event “HTH at the n th, $n + 1$ st and $n + 2$ st trial” in a Bernoulli sequence. Show that the sets $B_1, B_4, B_7, B_{10}, \dots$ are independent.
6. Let X be a set, \mathcal{R} a σ -field of subsets of X , and μ_1 and μ_2 measures on \mathcal{R} . Assume that $\mu_1(X) = \mu_2(X) < \infty$. Let \mathcal{L} be the family of those subsets $A \in \mathcal{R}$ for which $\mu_1(A) = \mu_2(A)$. Show that \mathcal{L} has the following properties:
 - (a) If $A, B \in \mathcal{L}$ and $B \subset A$, then $A \setminus B \in \mathcal{L}$.
 - (b) If $A_n \in \mathcal{L}, n = 1, 2, \dots$, are mutually disjoint, then $\cup_{n=1}^{\infty} A_n \in \mathcal{L}$.
7. Let (X, \mathcal{F}, μ) be a probability space (with \mathcal{F} a σ -field), and let $\{B_n\}_{n=1}^{\infty}$ be a countable collection of subsets from \mathcal{F} .
 - (a) Prove that

$$\mu(\limsup B_n) \geq \limsup \mu(B_n).$$
 - (b) Suppose that $X = [0, 1]$, \mathcal{F} is the field of Lebesgue measurable subsets of $[0, 1]$, and μ is Lebesgue measure. Show that if there exists a $\delta > 0$ such that $\mu(B_n) \geq \delta$ for every n , then there is at least one point which belongs to infinitely many B_n 's.