HOMEWORK 8. SOLUTIONS

Problem 2.1.1. Given a set X and subsets $A_1, A_2, A_3, \cdots$, let

$$A_+ = \limsup A_n = \bigcap_{k=1}^{\infty} \bigcup_{n \ge k} A_n$$

and

$$A_- = \liminf A_n = \bigcup_{k=1}^{\infty} \bigcap_{n \ge k} A_n.$$

Let f_+, f_- and f_n be the characteristic functions of the set A_+, A_- and A_n, respectively. Prove that $f_+ = \limsup f_n$ and $f_- = \liminf f_n$.

Solution. If $x \in A_+$, then $f_+(x) = 1$, and also for every $k > 0$ there exists $n \ge k$ such that $x \in A_n$, that is, such that $f_n(x) = 1$. This says that $\sup_{n \ge k} f_n(x) = 1$, and thus $\limsup f_n(x) = \lim_k \sup_{n \ge k} f_n(x) = 1$.

Conversely, if $x \notin A_+$, then $f_+(x) = 0$, and also that there exists $k_0 > 0$ such that $x \notin A_n$, for every $n \ge k_0$; that is, $f_n(x) = 0$ for every $n \ge k_0$. This implies that $\sup_{n \ge k_0} f_n(x) = 0$, hence that $\sup_{n \ge k} f_n(x) = 0$ for $k \ge k_0$. Thus $\limsup f_n(x) = 0$.

The case of $A_- = \liminf A_n$ can be worked out in a similar fashion. You can also apply a previous homework problem that says that $\liminf A_n = X \setminus \limsup(X \setminus A_n)$. Now $\chi_{X \setminus A_n} = 1 - \chi_{A_n}$. Therefore,

$$\chi_{\liminf A_n} = 1 - \limsup\bigl(1 - \chi_{A_n}\bigr) = 1 - 1 - \limsup\bigl(-\chi_{A_n}\bigr) = \liminf \chi_{A_n}.$$

Problem 2.1.3. (a) Let $f_n : X \to \mathbf{R}$ be a sequence of measurable functions. Show that the set $\bigl\{x \in X \bigm| \text{the sequence} f_n(x) \text{ converges}\bigr\}$ is measurable.

(b) Deduce that the set of points $\omega \in I$ for which the randomized series

$$\sum_{k=1}^{\infty} \frac{R_k(\omega)}{k}$$

converges, is a measurable subset of I.

Solution. (a) The sequence $\{f_n(x)\}$ converges if and only if it is a Cauchy sequence. That $\{f_n(x)\}$ is Cauchy means that for every positive integer r there exists an integer k such that $|f_n(x) - f_m(x)| < 1/r$, for all $n, m \ge k$.

Therefore, the set of points $x \in X$ such that $f_n(x)$ converges is

$$\bigcap_{r=1}^{\infty} \bigcup_{k=1}^{\infty} \bigcap_{n,m \ge k} E_{n,m,r}$$

where the set

$$E_{n,m,r} = \left\{ x \in X \,\middle|\, |f_n(x) - f_m(x)| < \frac{1}{r} \right\}$$

This set is measurable. Indeed, the difference of two measurable functions is measurable (Theorem 2.1.14 and Example 2.1.15), so $f_n - f_m$ is measurable. Furthermore, the absolute value of a measurable function is measurable (Theorem 2.1.12

and Example 2.1.13), so $|f_n - f_m|$ is measurable. Hence, by Proposition 2.1.2, the set $E_{n,m,r} = \{|f_n(x) - f_m(x)| < 1/r\}$ is measurable. Since $\mathcal{F}$ is a σ-field, the set $\bigcap_{n,m\geq k} E_{n,m,r}$ is measurable, and so is $\bigcup_k \bigcap_{n,m\geq k} E_{n,m,r}$, and so is $\bigcap_{r=1}^{\infty}\bigcup_{k=1}^{\infty}\bigcap_{n,m\geq k} E_{n,m,r}$, which is where $f_n(x)$ converges is therefore measurable.

(b) The series $\sum_k R_k(\omega)/k$ converges if and only if the sequence

$$f_n(\omega) = \sum_{k=1}^{n} \frac{R_k(\omega)}{k}$$

Thus (b) is an immediate consequence of (a).

Problem 2.1.8. Let $f : \mathbf{R} \to \mathbf{R}$ be monotone increasing. Show that f is measurable.

Solution. It has to be show that $f^{-1}(a, +\infty) = \{x \in \mathbf{R} \mid f(x) > a\}$ is a Borel set, for each $a \in \mathbf{R}$.

By definition, $f : \mathbf{R} \to \mathbf{R}$ is monotone increasing if $x > y$ implies $f(x) \geq f(y)$.

Given $a \in \mathbf{R}$, let

$$b = \inf\{x \in \mathbf{R} \mid f(x) > a\},$$

with the convention that the infimum of the empty set is $+\infty$.

Then

$$f^{-1}(a, +\infty) = \begin{cases} (b, +\infty) & \text{if } b \in \mathbf{R} \text{ and } f(b) = a \\ [b, +\infty) & \text{if } b \in \mathbf{R} \text{ and } f(b) > a \\ \emptyset & \text{if } b = +\infty \\ (-\infty, +\infty) & \text{if } b = -\infty \end{cases}$$

If $b = +\infty$, then (by the convention stated above) the set $\{f(x) > a\}$ is empty, and conversely.

If $b = -\infty$, then $f(x) > a$ for every $x \in \mathbf{R}$, and so $f^{-1}(a, +\infty) = (-\infty + \infty)$.

If $b \in \mathbf{R}$ and $f(b) = a$, then $f(x) > f(b) = a$ for $x > b$ because f is monotone increasing and the definition of b.

If $b \in \mathbf{R}$ and $f(b) > a$, then $f(x) \geq f(b) > a$ if $x \geq b$, because f is monotone increasing and definition of b.