MATH 650. HOMEWORK 7. SOLUTIONS

Problem 1. Let $(X, \mathcal{F}, \mu)$ be a probability space. Let $A_1, A_2, \ldots$ be a sequence of subsets of X belonging to $\mathcal{F}$.

(a) Show that if $A_1 \supset A_2 \supset A_3 \supset \cdots$, then

$$\mu(\bigcap_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n).$$

(b) Show that if $A_1 \subset A_2 \subset A_3 \subset \cdots$, then

$$\mu(\bigcup_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n).$$

Solution (a) The sets $B_1 = A_1 \setminus A_2, B_2 = A_2 \setminus A_3, \ldots$ are disjoint, belong to $\mathcal{F}$, and

$$\bigcup_{n=1}^{\infty} B_n = A_1 \setminus \bigcap_{n=1}^{\infty} A_n.$$

To see this identity, let $x \in \bigcup_{n=1}^{\infty} B_n$. Then $x \in B_n$ for at least one index n. Thus $x \in A_n \subset A_1$ and $x \notin A_{n+1} \supset \bigcap_{n=1}^{\infty} A_n$. The reverse containment: if $x \in A_1 \setminus \bigcap_{n=1}^{\infty} A_n$, then $x \notin \bigcap_{n=1}^{\infty} A_n$, and thus there is a smallest index $n \geq 2$ (because $x \in A_1$) such that $x \notin A_n$. So $x \in A_{n-1} \setminus A_n = B_{n-1}$. Thus it follows that

$$\mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \mu(A_1) - \lim_{n\to\infty} \mu(A_n). \tag{0.1}$$

On the other hand

$$\begin{aligned}
\mu\left(\bigcup_{n=1}^{\infty} B_n\right) &= \sum_{n=1}^{\infty} \mu(B_n) && \text{(because the } B_n \text{ are disjoint and } \mu \text{ countably additive)} \\
&= \sum_{n=1}^{\infty} (\mu(A_n) - \mu(A_{n+1})) && \text{(because } B_n = A_n \setminus A_{n+1} \text{ and } \mu \text{ is additive)} \\
&= \mu(A_1) - \lim_{n\to\infty} \mu(A_n) && \text{(by calculus of series),}
\end{aligned}$$

which together with Equation (1) and the fact that μ is a probability measure (why?) gives (a).

(b) Let $B_n = X \setminus A_n$. Then $B_1 \supset B_2 \supset \cdots$, as in (a). Thus

$$\mu(\bigcap_{n=1}^{\infty} B_n = \lim_{n\to\infty} \mu(B_n).$$

But $\bigcap_{n=1}^{\infty} B_n = X \setminus \bigcup_{n=1}^{\infty} A_n$ and $\mu(B_n) = \mu(X \setminus A_n) = \mu(X) - \mu(A_n) = 1 - \mu(A_n)$, and (b) obtains.

Problem 2. Let X be a probability space and A_n measurable sets. Show that the probability of $\liminf_n A_n^c$ is 0 if and only if the probability of $\limsup_n A_n$ is 1.

Solution By definition

$$\liminf_n A_n^c = \bigcup_{k=1}^{\infty} \bigcap_{n>k} A_n^c$$

and

$$\limsup_n A_n = \bigcap_{k=1}^{\infty} \bigcup_{n>k} A_n$$

Thus taking complements

$$(\limsup_n A_n)^c = X \setminus \left(\bigcap_{k=1}^{\infty} \bigcup_{n>k} A_n \right) = \bigcup_{k=1}^{\infty} \bigcap_{n>k} (X \setminus A_n) = \liminf_n A_n^c$$

and since $m(X) = 1$ it follows that

$$1 - \mu(\limsup_n A_n) = \liminf_n A_n^c.$$

Problem 3. Let $(X, \mathcal{F}, \mu)$ be a probability space, and let $A_1, A_2, \ldots$ be in $\mathcal{F}$. Show that

$$\mu(\liminf_n A_n) \leq \liminf_n \mu(A_n) \leq \limsup_n \mu(A_n) \leq \mu(\limsup_n A_n).$$

Solution If x_n is a sequence of real numbers then

$$\limsup_n x_n = \lim_k \left(\sup_{n>k} x_n \right) \qquad \text{and} \qquad \liminf_n x_n = \lim_k \left(\inf_{n>k} x_n \right)$$

so $\limsup_n x_n \geq \liminf_n x_n$ and the middle inequality follows.

To prove the first inequality, let $B_k = \bigcap_{n>k} A_n$. Then $B_1 \subset B_2 \subset \cdots$, and $\bigcup_{k=1}^{\infty} B_k = \liminf_n A_n$. Since X is a probability space, Problem 1 applies:

$$\mu \left(\bigcup_{k=1}^{\infty} B_k \right) = \lim_k \mu(B_k)$$

Moreover, $B_k \subset A_n$ for every $n > k$, so $\mu(B_k) \leq \inf_{n>k} \mu(A_n)$. Hence, by definition of $\liminf$, it obtains $\lim_k \mu(B_k) \leq \liminf_n \mu(A_n)$.

The third inequality is proved similarly.

Problem 4. Let $(X, \mathcal{F}, \mu)$ be a probability space. Show that if $A_1, A_2, \cdots, A_n$ are independent sets from $\mathcal{F}$, then the sets $A_1^c, A_2, \cdots, A_n$ are also independent.

Solution It suffices to show that for any sequence of integers $2 \leq i_1 < i_2 \cdots < i_k \leq n$, we have

$$\mu(A_1^c \cap A_{i_1} \cap \cdots \cap A_{i_k}) = \mu(A_1^c)\mu(A_{i_1}) \cdots \mu(A_{i_k})$$

Now, since $A_1 \cup A_1^c = X$,

$$\begin{aligned} A_{i_1} \cap \cdots \cap A_{i_k} &= (A_1^c \cup A_1) \cap A_{i_1} \cap \cdots \cap A_{i_k} \\ &= (A_1^c \cap A_{i_1} \cap \cdots \cap A_{i_k}) \bigcup (A_1 \cap A_{i_1} \cap \cdots \cap A_{i_k}) \end{aligned}$$

is a disjoint union, so that taking measures

$$\mu(A_{i_1} \cap \cdots \cap A_{i_k}) = \mu \left(A_1^c \cap A_{i_1} \cap \cdots \cap A_{i_k} \right) + \mu \left(A_1 \cap A_{i_1} \cap \cdots \cap A_{i_k} \right)$$

Since the sets A_k are independent, this can be written

$$\mu(A_{i_1}) \cdots \mu(A_{i_k}) = \mu(A_1^c \cap A_{i_1} \cap \cdots \cap A_{i_k}) + \mu(A_1)\mu(A_{i_1}) \cdots \mu(A_{i_k})$$

and so

$$\begin{aligned}
\mu(A_1^c \cap A_{i_1} \cap \cdots \cap A_{i_k}) &= (\mu(A_{i_1}) \cdots \mu(A_{i_k})) - (\mu(A_1)\mu(A_{i_1}) \cdots \mu(A_{i_k})) \\
&= (1 - \mu(A_1))\mu(A_{i_1}) \cdots \mu(A_{i_k}) \\
&= \mu(A_1^c)\mu(A_{i_1}) \cdots \mu(A_{i_k})
\end{aligned}$$