MATH 650. HOMEWORK 6. SOLUTIONS

Problem 1. Let A be a measurable subset of $\mathbf{R}$. The density of A is well defined if the limit

$$D(A) = \lim_{T\to\infty} \frac{\mu_L(A \cap [-T,T])}{2T}$$

exists.

(1) Find a measurable set whose density is well defined.

(2) Show that if A and B have well-defined density and are disjoint, then $A \cup B$ has a well-defined density and

$$D(A \cap B) = D(A) + D(B)$$

(3) Show that there exists sets A and A_n, $n = 1, 2, \cdots$, with well-defined densities such that $A = \bigcup_{n=1}^{\infty} A_n$ (disjoint union) but

$$D(A) \neq \sum_{n=1}^{\infty} D(A_n).$$

Solution. (1) Such A is made of disjoint longer and longer intervals, with bigger and bigger gaps between them. This way the ratio

$$\frac{\mu_L(A \cap [-T,T])}{2T}$$

will oscillate as $T \to \infty$, and the limit will fail to exist.

Take $A = (1,2) \cup (2^2, 2^3) \cup (2^4, 2^5) \cup \cdots = \bigcup_{n=0}^{\infty} (2^{2n}, 2^{2n+1})$ For $T = 2^{2k}$, $k = 1, 2, \cdots$, we have

$$A \cap [-2^{2k}, 2^{2k}] = (1,2) \cup (2^2, 2^3) \cup \cdots \cup (2^{2k-2}, 2^{2k-1})$$

so

$$\mu(A \cap [-2^{2k}, 2^{2k}]) = 1 + 2^2 + \cdots + 2^{2k-1} = \frac{4^k - 1}{3}$$

and

$$\lim_{k\to\infty} \frac{\mu(A \cap [-2^{2k}, 2^{2k}])}{2(4^k)} = \frac{1}{6}.$$

For $T = 2^{2k+1}$ we have

$$A \cap [-2^{2k+1}, 2^{2k+1}] = (1,2) \cup (2^2, 2^3) \cup \cdots \cup (2^{2k-2}, 2^{2k-1}) \cup (2^{2k}, 2^{2k+1})$$

so

$$\mu(A \cap [-2^{2k+1}, 2^{2k+}]) = 1 + 2^2 + \cdots + 2^{2k} = \frac{4^{k+1} - 1}{3}$$

and

$$\lim_{k\to\infty} \frac{\mu(A \cap [-2^{2k+1}, 2^{2k+1}])}{2(2^{2k+1})} = \frac{1}{3}.$$

(2) This is an easy consequence of the properties of limits.

(3) Take $A_n = [n, n+1]$. Then $A = \bigcup_{n=1}^{\infty} A_n = [1, \infty)$. Computing densities

$$D(A_n) = \lim_{T\to\infty} \frac{\mu(A_n \cap [-T,T])}{2T} = \lim_{T\to\infty} \frac{1}{2T} = 0$$

and

$$D(A) = \lim_{T\to\infty} \frac{\mu(A\cap[-T,T])}{2T} = \lim_{T\to\infty} \frac{T-1}{2T} = \frac{1}{2}$$

Problem 2. Let X be an uncountable set. Let $\mathcal{R}$ be the collection of all finite subsets of X. Given $A \in \mathcal{R}$ let $\mu(A)$ be the number of elements in A.

(1) Show that $\mathcal{R}$ is a ring and that μ is a measure on $\mathcal{R}$.
(2) Identify μ^*
(3) What are $\mathcal{M}$ and $\mathcal{M}_F$?
(4) Is every subset of X measurable?
(5) $(X, \mathcal{M}, \mu^*)$ is not σ-finite.

Solution. (1) was part of another homework.

(2) By definition, if $A \subset X$, $\mu^*(A)$ is the infimum of the approximate outer-measures for A. Then

$$\mu^*(A) = \begin{cases} \mu(A) & \text{if } A \in \mathcal{R}, \\ \infty & \text{if } A \notin \mathcal{R} \end{cases}$$

(3) By definition, $B \in \mathcal{M}_F$ is there exists a sequence of sets $A_n \in \mathcal{R}$ such that $\mu^*(B\Delta A_n) \to 0$ as $n \to \infty$. Since $A_n \in \mathcal{R}$, the sets are finite. Moreover, if $\mu^*(B\Delta A_n) \to 0$, then $B\Delta A_n$ must be a finite set for all but finitely many indexes n. Since we can write $B \subset (B\Delta A_n) \cup (B \cap A_n)$, we deduce that B itself must be also a finite set. It follows that $\mathcal{M}_F = \mathcal{R}$.

To describe $\mathcal{M}$ we note that by definition a set $A \in \mathcal{M}$ is there is a sequence of sets $A_n \in \mathcal{M}_F$ such that $A = \bigcup_{n=1}^{\infty} A_n$. By the above each A_n is a finite sets. Hence A is a countable union of finite sets and thus a countable sets. That is $\mathcal{M}$ is the collection of countable subsets of X.

(4) Since X is itself uncountable, $X \notin \mathcal{M}$, that is X is not measurable.

(5) Same as above, X is not σ-finite because it is uncountable and so it cannot be written as a countable union of finite sets.

Problem 3. Show that given any $\delta > 0$ there exists an open dense subset U of $\mathbf{R}$ with Lebesgue measure $\mu_L(U) < \delta$.

Solution. We know that the set of rational numbers $\mathbf{Q}$ is countable and dense in $\mathbf{R}$. Enumerate the rational numbers in a sequence $\mathbf{Q} = \{q_1, q_2, q_3, \cdots\}$ and take

$$U = \bigcup_{n=1}^{\infty} (q_n - \delta/2^{n+2}, q_n + \delta/2^{n+2})$$

Each $(q_n - \delta/2^{n+2}, q_n + \delta/2^{n+2})$ is an open interval (of length $\delta/2^{n+1}$), so U is open because is a union of open sets.

Furthermore $\mathbf{Q} \subset U$, so U is also dense in $\mathbf{R}$.

Because μ_L is countably subbaditive,

$$\mu(U) \leq \sum_{n=1}^{\infty} \mu_L((q_n - \delta/2^{n+2}, q_n + \delta/2^{n+2})) = \sum_{n=1}^{\infty} \frac{\delta}{2^{n+1}} = \frac{\delta}{2} < \delta.$$