MATH 650. HOMEWORK 5. SOLUTIONS

Problem 1. *Let X be an uncountable set. Let $\mathcal{R}$ be the collection of all the finite subsets of X. Given $A \in \mathcal{R}$ let $\mu(A)$ be the number of elements in A. Show that $\mathcal{R}$ is a ring and μ is a measure on $\mathcal{R}$.*

Solution. If A and B are in $\mathcal{R}$, they are finite sets, and so is $A \cup B$ and $A - B$.

If $A, B \in \mathcal{R}$ are disjoint, then the number of elements in $A \cup B$ equals the number of elements in A plus the number of elements in B, so $\mu(A \cup B) = \mu(A) + \mu(B)$.

For μ to be a measure, it must be countably additive. Let then $\{A_i\}_{i=1}^{\infty} \subset \mathcal{R}$ be a countable collection of mutually disjoint sets such that $A = \bigcup_{i=1}^{\infty} A_i$ is also in $\mathcal{R}$. Then A and all A_i are finite, and so only finitely many A_i are non-empty. Therefore $\sum_{i=1}^{\infty} \mu(A_i)$ is a finite sum and

$$\mu(A) = \sum_{i=1}^{\infty} \mu(A_i)$$

by additivity of μ. □

Problem 2. *Let X be an infinite set and $\mathcal{R}$ the collection of all countable subsets of X.*

1. *Is $\mathcal{R}$ a σ-ring?*
2. *Let μ be a measure on $\mathcal{R}$. Show that there is a function $f : X \to [0, \infty)$ such that*
$$\mu(A) = \sum_{x \in A} f(x),$$
for all $A \in \mathcal{R}$.
3. *Show that f in (2) has the following properties: the set $\{x \in X \mid f(x) \neq 0\}$ is countable, and $\sum f(x) < \infty$.*
4. *Conversely, show that if $f : X \to [0, \infty)$ has these two properties, then the formula*
$$\mu(A) = \sum_{x \in A} f(x)$$
defines a measure on $\mathcal{R}$.

Solution. (1) Yes, a countable union of countable sets is a countable set.
(2) Each one-point set $\{x\} \in \mathcal{R}$. If $A \in \mathcal{R}$, A is countable, and so it can be written as a countable disjoint union of one-point sets: $A = \bigcup_{x \in A} \{x\}$. Thus, setting $f(x) = \mu(\{x\})$, the countably additivity of μ gives

$$\mu(A) = \sum_{x \in A} f(x).$$

(3) The set $\{f(x) > 0\}$ is countable. Indeed, let A_n be the interval $A_n = (\frac{1}{n+1}, \frac{1}{n}]$, $n = 1, 2, \cdots$ and $A_0 = (1, \infty)$. Then $\{A_n\}_{n=0}^{\infty}$ is a countable partition of $(0, \infty)$, and so $B_n = \{x \in X \mid f(x) \in A_n\}$ is a countable partition of $\{f(x) > 0\}$. If $\{f(x) > 0\}$ was uncountable, then at least one of the B_n must be also uncountable (for otherwise $\{f(x) > 0\}$ would be a countable union of countable sets, hence

countable). Suppose then that B_{n_0} is uncountable. Then it contains an infinite countable set $C = \{x_1, x_2, \cdots\}$. Since C is countable, $C \in \mathcal{R}$. But its measure

$$\mu(C) = \sum_{i=1}^{\infty} f(x_i) \geq \sum_{i=1}^{\infty} \frac{1}{n_0 + 1} = \infty,$$

contradicting the fact that $\mu(A)$ is a non-negative number for every $A \in \mathcal{R}$.

Once we know that $Y = \{f(x) > 0\}$ is countable, we know that $Y \in \mathcal{R}$ and so $0 \leq \mu(Y) < \infty$ and

$$\sum_{x \in X} f(x) = \sum_{f(x)>0} f(x) = \sum_{x \in Y} f(x) = \mu(Y) < \infty.$$

□

Problem 3. *Let X be the real line and $\mathcal{R} = \mathcal{R}_{Leb}$. Given $A \in \mathcal{R}$, let $\mu(A) = 1$ if, for some positive ε, A contains the interval $(0, \varepsilon)$; otherwise let $\mu(A) = 0$.*

Show that μ is an additive set function, but it is not countably additive.

Solution. We have to show that if $A, B \in \mathcal{R}$ are disjoint, then $\mu(A \cup B) = \mu(A) + \mu(B)$. A set in $\mathcal{R}$ is a finite union of intervals.

Let $A = A_1 \cup \cdots \cup A_m$ and $B = B_1 \cup \cdots \cup B_n$, where the A_i's and B_j's are intervals. Assume that they are ordered so that the right endpoint of A_i is less than or equal to the left endpoint of A_{i+1}, $i = 1, \cdots, m-1$, and similarly for the B_j's. Also assume that the right endpoint of A_1 is smaller that the left endpoint of B_1. (You should verify that this can be done.)

There are two cases to consider.

1. $\mu(A \cup B) = 1$. This means that there exists $\varepsilon > 0$ such that $(0, \varepsilon) \subset A \cup B = A_1 \cup \cdots \cup A_m \cup B_1 \cup \cdots B_n$. But given that A_1 is the left-most interval in $A \cup B$, this implies that there exists $a > 0$ such that $(0, a) \subset A_1$, and hence also that $B \cap (0, a) = \emptyset$. It follows that $\mu(A) = 1$ and $\mu(B) = 0$.
2. $\mu(A \cup B) = 0$. This means that $A \cup B$ does contains no interval of the form $(0, \varepsilon)$, and since $A, B \subset A \cup B$, the same must be true for A and for B. Thus $\mu(A) = \mu(B) = 0$.

To show that μ is not countably additive, let $A_n = (\frac{1}{n+1}, \frac{1}{n}]$, $n = 1, 2, \cdots$, so that $\mu(A_n) = 0$ for all n. Then $A = \bigcup_{i=1}^{\infty} (\frac{1}{n+1}, \frac{1}{n}] = (0, 1]$ is in $\mathcal{R}$, but

$$\mu(A) = 1 \neq \sum_{i=1}^{\infty} \mu(A_i) = 0 + 0 + 0 + \cdots = 0.$$

□

Problem 4. *Given any collection $\mathcal{C}$ of subsets of X, show that there is a smallest ring $\mathcal{R}$ containing $\mathcal{C}$.*

Solution. There is always a ring which contains $\mathcal{C}$, namely the power set ring 2^X. Let $\mathcal{R}$ be the intersection of all the rings which contain $\mathcal{C}$. Then $\mathcal{R}$ is a non-empty collection of subsets of X.

It is also a ring. For if A, B are in $\mathcal{R}$, then they are in every ring which contains $\mathcal{C}$. If $\mathcal{S}$ is a ring which contains $\mathcal{C}$, then the union $A \cap B$ and the difference $A - B$ are also in $\mathcal{S}$. Thus $A \cap B$ and $A - B$ are in every ring which contains $\mathcal{C}$, that is, they are in $\mathcal{R}$. □