MATH 650. HOMEWORK 5. DUE 9/19/02

Problem 1. *Let X be an uncountable set. Let $\mathcal{R}$ be the collection of all the finite subsets of X. Given $A \in \mathcal{R}$ let $\mu(A)$ be the number of elements in A. Show that $\mathcal{R}$ is a ring and μ is a measure on $\mathcal{R}$.*

Problem 2. *Let X be an infinite set and $\mathcal{R}$ the collection of all countable subsets of X.*

(1) *Is $\mathcal{R}$ a σ-ring?*
(2) *Let μ be a measure on $\mathcal{R}$. Show that there is a function $f : X \to [0, \infty)$ such that*
$$\mu(A) = \sum_{x \in A} f(x),$$
for all $A \in \mathcal{R}$.
(3) *Show that f in (2) has the following properties: the set $\{x \in X \mid f(x) \neq 0\}$ is countable, and $\sum f(x) < \infty$.*
(4) *Conversely, show that if $f : X \to [0, \infty)$ has these two properties, then the formula*
$$\mu(A) = \sum_{x \in A} f(x)$$
defines a measure on $\mathcal{R}$.

Problem 3. *Let X be the real line and $\mathcal{R} = \mathcal{R}_{Leb}$. Given $A \in \mathcal{R}$, let $\mu(A) = 1$ if, for some positive ε, A contains the interval $(0, \varepsilon)$; otherwise let $\mu(A) = 0$.*

Show that μ is an additive set function, but it is not countably additive.

Problem 4. *Given any collection $\mathcal{C}$ of subsets of X, show that there is a smallest ring $\mathcal{R}$ containing $\mathcal{C}$.*