

## MATH 650. HOMEWORK 4. SOLUTIONS

1. Let  $X$  be an infinite set and let  $\mathcal{R}$  be the following collection of subsets:  $A \in \mathcal{R}$  if and only if  $A$  is finite or  $A^c$  is finite. Let  $\mu$  be the following function on  $\mathcal{R}$ :  $\mu(A) = 1$  if  $A$  is finite, and  $\mu(A) = 0$  if  $A^c$  is finite. Is  $\mu$  a measure?

**Solution.**

You should check that  $\mathcal{R}$  is a ring.

There are two cases to consider: (1)  $X$  is countable, and (2)  $X$  is uncountable.

**Case (1)** If  $X$  is countable, then  $\mu$  is not a measure. Indeed, write  $X = \{x_0, x_1, x_2, \dots\}$ . Then the sets  $A_i = \{x_i\}$ ,  $i = 1, 2, \dots$  belong to  $\mathcal{R}$ , are disjoint, and  $\mu(A_i) = 0$ . Furthermore,  $A = \bigcup_{i=1}^{\infty} A_i = \{x_1, \dots\}$  is such that  $A^c = \{x_0\}$  is finite, so also  $A \in \mathcal{R}$ . But  $\mu(A) = 1$ . Thus

$$\mu(A) = 1 \neq \sum_{i=1}^{\infty} \mu(A_i) = 0.$$

**Case (2)** If  $X$  is uncountable, then  $\mu$  is a measure. The key observation is that **if  $A, B \in \mathcal{R}$ , and both are infinite, then they must intersect.**

Let  $\{A_i\}_{i=1}^{\infty}$  be a disjoint collection of elements of  $\mathcal{R}$  such that  $A = \bigcup_{i=1}^{\infty} A_i \in \mathcal{R}$ .

There are now two cases to consider: (2a)  $A$  finite, and (2b)  $A^c$  finite.

(2a) If  $A$  is finite, then all the  $A_i$  are also finite because  $A_i \subset A$ . Thus  $\mu(A) = 0 = \mu(A_i)$ .

(2b) If  $A^c$  is finite then  $A$  is infinite. In this situation, not all the  $A_i$  can be finite, for if they were, then  $A$  would be a countable union of finite sets, hence countable. But then  $X = A \cup A^c$  would be the union of a countable set and a finite set, hence also countable.

Hence at least one of the  $A_i$  is infinite. But in fact exactly one of them is infinite because of the key observation above. Thus in this case also

$$\mu(A) = 1 = \sum_{i=1}^{\infty} \mu(A_i) = 1 + 0 + 0 + 0 \dots = 1.$$