Math 650. Homework 12. Solutions

Exercise 3.1.1. Let I be the unit interval $0 \leq x \leq 1$, and let $I_{k,n}$ be the subinterval

$$\frac{k}{n} \leq x \leq \frac{k+1}{n} \qquad 0 \leq k \leq n.$$

Let f_1 be the characteristic function of $I_{0,1}$, f_2 and f_3 the characteristic functions of $I_{0,2}$ and $I_{1,2}$, and so on. Show that the sequence f_n converges to 0 in $\mathcal{L}^1(I)$ but it does not converge pointwise anywhere.

Solution. A figure will help you to understand what is going on.

Given an integer $n = 1, 2, \cdots$ there is a unique integer $m = 1, 2, \cdots$ such that

$$\frac{m(m-1)}{2} + 1 \leq n < \frac{m(m+1)}{2} + 1$$

and so the interval $I_{k,n}$, $(0 \leq k < n)$ has length $1/m$. Thus the integral

$$\int_I |f_n| \cdot \mu_L = \frac{1}{m} \leq \frac{2}{\sqrt{n}}$$

which converges to 0 as $n \to \infty$.

For $x \in I$, the sequence of values $f_n(x)$ contains infinitely many 0's and infinitely many 1's and thus it cannot converge. □

Exercise 3.1.3. In Exercise 3.1.1 above, extract a subsequence of f_n which converges to 0 almost everywhere.

Solution. Take $g_n = f_{(n(n-1)/2)+1}$. □

Exercise 3.1.2. Let f_n be the function on $(0, 1]$ that is equal to 0 in $[1/n, 1]$ and equal to n in $(0, 1/n)$. Show that f_n converges pointwise to 0 everywhere in $(0, 1]$ as $n \to \infty$, but it does not converge in $\mathcal{L}^1$.

Solution. The function f_n can be expressed as

$$f_n = n\chi_{(0,1/n)}.$$

Therefore, given $x \in (0, 1]$, if $n > 1/x$, then $f_n(x) = 0$. It follows that $\lim_{n\to\infty} f_n(x) = 0$ for all $x \in (0, 1]$.

To show that f_n does not converge in $\mathcal{L}^1$ we will show that it is not a Cauchy sequence there. Let n, m be integers with $n < m$. Then the difference

$$f_m - f_n = (m - n)\chi_{(0,1/m)} - n\chi_{[1/m,1/n)}$$

and thus

$$|f_m - f_n| = (m-n)\chi_{(0,1/m)} + n\chi_{[1/m,1/n)}.$$

Therefore, the $\mathcal{L}^1$-norm

$$\begin{aligned} \|f_m - f_n\|_1 &= \int_{(0,1]} |f_m - f_n| \cdot \mu_L \\ &= 2\frac{m-n}{m} \end{aligned}$$

which does not converge to 0 as $n \to \infty$ and $m \to \infty$ (take $m = 2n \to \infty$). □

Exercise 3.2.2. Let V be an inner product space. Show that

$$\|v+w\|^2 + \|v-w\|^2 = 2\|v\|^2 + 2\|w\|^2.$$

Solution. This is a calculation. Compute each term on the left side:

$$\begin{aligned} \|v+w\|^2 + &= \langle v+w, v+w \rangle \\ &= \langle v,v \rangle + \langle w,w \rangle + \langle v,w \rangle + \langle w,v \rangle \\ &= \|v\|^2 + \|w\|^2 + \langle v,w \rangle + \langle w,v \rangle \end{aligned}$$

and

$$\begin{aligned} \|v-w\|^2 + &= \langle v-w, v-w \rangle \\ &= \langle v,v \rangle + \langle w,w \rangle - \langle v,w \rangle - \langle w,v \rangle \\ &= \|v\|^2 + \|w\|^2 - \langle v,w \rangle - \langle w,v \rangle \end{aligned}$$

and add them up. □

Exercise 3.2.4. Let $X = (0,1]$ equipped with Lebesgue measure μ. Show that the function $f(x) = x^{-3/4}$ is in $calL^1(X,\mu)$ but not in $\mathcal{L}^2(X,\mu)$.

Solution. Use Exercise 2.4.1. The functions f and f^2 are nonnegative and measurable on $J = (0,1]$, and Riemann integrable on $[a,1]$ for every $0 < a < 1$. The Riemann integrals are

$$\int_a^1 f(x) \cdot dx = \int_a^1 x^{-3/4} \cdot dx = 4 - 4a^{1/4}$$

and

$$\int_a^1 f(x) \cdot dx = \int_a^1 x^{-3/4} \cdot dx = -2 + \frac{2}{a^{1/2}}$$

Therefore, by Exercise 2.4.1,

$$\int_J f \cdot \mu = \lim_{a \to 0} \left(4 - 4a^{1/4}\right) = 4 < \infty$$

and

$$\int_J f \cdot \mu = \lim_{a \to 0} \left(-2 + \frac{2}{a^{1/2}}\right) = \infty.$$

□