Homework 11. Solutions

Problem 2.3.2. Let $f_n : \mathbf{R} \to \mathbf{R}$ be $1/n$ times the characteristic function of the interval $(0, n)$. Show that $f_n \to 0$ uniformly and $\int f_n \cdot \mu_L = 1$. Why isn't it a counterexample to the Lebesgue Dominated Convergence theorem.

Solution. Uniform convergence. Given $\epsilon > 0$, let N be an integer greater than $1/\epsilon$. If $n > N$, then

$$\sup_{x \in \mathbf{R}} |f_n(x)| \le \frac{1}{n} < \epsilon.$$

Thus $\lim_{n\to\infty} \sup_x |f_n(x)| = 0$, which is what uniform convergence means.

The function $f_n = (1/n)\chi_{(0,n)}$ is a simple function and

$$\int \frac{1}{n}\chi_{(0,n)} \cdot \mu_L = \frac{1}{n}\mu_L((0,n)) = 1.$$

This example does not contradict the Lebesgue Dominated Convergence theorem because there is no integrable function g such that $g \ge f_n$ for all n. Indeed, assume that $g \ge f_n$ for all n. Then for each k, the sum $s_k = \sum_{n=1}^{k}(1/n)\chi_{(n-1,n)}$ is a simple function with $g \ge s_k \ge 0$, because $s_k = f_n$ on $(n-1, n)$, for all $1 \le n \le k$. Hence

$$\int g \cdot \mu_L \ge \int s_k \cdot \mu_L$$

for all k. But the integral

$$\int s_k \cdot \mu_L = \sum_{n=1}^{k} \frac{1}{n},$$

which diverges to ∞ as $k \to \infty$. Hence $\int g \cdot \mu_L = \infty$. $\square$

Problem 2.3.5. Let $(X, \mathcal{F}, \mu)$ be a measure space. A measurable function $f : X \to \mathbf{R}$ is square integrable if $\int |f|^2 < \infty$. Show that if $\mu(X) < \infty$, then every square-integrable function is integrable.

Solution. Let $A = \{|f| > 1\}$ and $B = \{|f| \leq 1\}$. Then A and B are disjoint measurable sets with $X = A \cup B$. Then

$$\begin{aligned}
\int_X |f| &= \int_A |f| + \int_B |f| && \text{because } A \cup B = X \text{ and } A \cap B = \emptyset \\
&\leq \int_A |f|^2 + \int_B 1 && \text{because } |f| < |f|^2 \text{ on } A \text{ and } |f| \leq 1 \text{ on } B \\
&\leq \int_X |f|^2 + \int_X 1 && \text{because } A, B \subset X \text{ and } |f|^2, 1 \geq 0 \\
&< \infty \quad \text{by the hypotheses } \textstyle\int_X |f|^2 < \infty \text{ and } \mu(X) < \infty
\end{aligned}$$

$\square$

Problem 2.3.8. Let $(X, \mathcal{F}, \mu)$ be a measure space. A sequence, f_n, of measurable functions is said to converge in measure to 0 if, for all $\epsilon > 0$,

$$\lim \mu\{|f_n| > \epsilon\} = 0$$

Show that if $\int f_n \cdot \mu \to 0$ as $n \to \infty$, then f_n converges to 0 in measure, and show that the converse is not true.

Solution. By Chebyshev's inequality,

$$\mu\{|f_n| > \epsilon\} \leq \frac{1}{\epsilon} \int |f_n| \cdot \mu.$$

Now take the limit as $n \to \infty$.

For the converse, let $X = \mathbf{R}$, μ Lebesgue measure. Take $f_n = (1/n)\chi_{(0,n)}$. Then $\lim \int |f_n| = 1$. On the other hand, given $\epsilon > 0$, if $n > 1/\epsilon$, then

$$\mu\{|f_n| > \epsilon\} = 0,$$

so that f_n converges to 0 in measure. $\square$

Problem 2.3.12. Let $(X, \mathcal{F}, \mu)$ be a measure space and f a non-negative measurable function on X. For $A \in \mathcal{F}$ let

$$\mu_f(A) = \int_A f \cdot \mu.$$

Show that μ_f is a measure on $\mathcal{F}$. Show that if $g \geq 0$ is a measurable function, then

$$\int_E f \cdot \mu_f = \int_E gf \cdot \mu,$$

for all $E \in \mathcal{F}$.

Solution. It is clear that $\mu_f(A) \geq 0$ because $f \geq 0$. To show countable additivity, use Theorem 11, page 67. If $A_1, A_2, \cdots$ is a countable collection of disjoint measurable sets, and if $A = \bigcup_{n=1}^{\infty} A_n$, then

$$\mu_f(A) = \int_A f \cdot \mu = \sum_{n=1}^{\infty} \int_{A_n} f \cdot \mu = \sum_{n=1}^{\infty} \mu_A(A).$$

For the second part, suppose first that $g = \chi_A$, the characteristic function of a measurable set $A \in \mathcal{F}$. Then

$$\int_E \chi_A \mu_f = \mu_f(E \cap A) = \int_{E \cap A} f\mu = \int_E \chi_A \cdot \mu.$$

It follows that

$$\int_E g \cdot \mu_f = \int_E gf\mu$$

for every simple function $g \geq 0$.

In general, if $g \geq 0$ is measurable, let s_n be a sequence of simple functions, $0 \leq s_n \leq g$, which increase pointwise to g. Then the monotone convergence theorem implies that

$$\int_E g \cdot \mu_f = \lim \int_E s_n \cdot \mu_f$$

Also, $s_n f$ increases pointwise to gf, so this same theorem implies

$$\int_E gf \cdot \mu = \lim \int_E s_n f \cdot \mu.$$

The stated identity now follows because for simple nonnegative functions it was just proved that

$$\int_E s_n \cdot \mu_f = \int_E s_n f \cdot \mu.$$

□

Problem 2.5.1. Done in class.

Problem 2.5.8. Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space, $f : X \to \mathbf{R}$ a nonnegative measurable function. Let

$$A_f = \{(x,t) \in X \times \mathbf{R} \mid 0 \leq t \leq f(x)\}$$

Show that A_f is a measurable subset of $X \times \mathbf{R}$ and that

$$(\mu \otimes \mu_L)(A_f) = \int_X f \cdot \mu$$

where μ_L is Lebesgue measure on $\mathbf{R}$.

Solution. Let $F : X \times \mathbf{R} \to \mathbf{R}$ be the map $F(x,t) = f(x) - t$. This map is measurable because it is the composition of the maps $G : (x,t) \mapsto (f(x), t)$ and $H : (z,t) \mapsto z - t$. The map H is measurable (Example 2.1.15), in fact it is continuous.

If $A \times B$ is a product set in $\mathbf{R} \times \mathbf{R}$, then

$$G^{-1}(A \times B) = \{(x,t) \mid G(x,t) = (f(x), t) \in A \times B\} = f^{-1}(A) \times B$$

is a product set in $X \times \mathbf{R}$.

To show that F is measurable it must be shown that if B is a Borel subset of $\mathbf{R}$, then $F^{-1}(B) = \{F(x,t) \in B\}$ is a Borel set. In fact it suffices to consider B an open subset of $\mathbf{R}$. Because $F = H \circ G$, the inverse image $F^{-1}(B) = (H \circ G)^{-1}(B) = G^{-1}(H^{-1}(B))$. Since H is continuous and B is open, the inverse image $H^{-1}(B)$ can be written as a countable union of open subsets of $\mathbf{R} \times \mathbf{R}$ of the form $U \times V$, where U, V are open in $\mathbf{R}$. Thus there are countable many open subsets $U_n, V_n \subset \mathbf{R}$ such that

$$H^{-1}(B) = \bigcup_{n=1}^{\infty} U_n \times B_n$$

Therefore

$$F^{-1}(B) = G^{-1}(\bigcup_{n=1}^{\infty} U_n \times B_n) = \bigcup_{n=1}^{\infty} G^{-1}(U_n \times V_n) = \bigcup_{n=1}^{\infty} f^{-1}(U_n) \times V_n$$

. But $f^{-1}(U_n)$ is measurable and so $f^{-1}(U_n) \times V_n$ is a product set in $X \times \mathbf{R}$. Hence $F^{-1}(B)$ is a countable union of product sets, thus an element of the σ-field $\mathcal{F} \times \mathcal{B}_\infty$.

Furthermore, the set

$$A_f = \{(x,t) \mid 0 \le t \le f(x)\} = F^{-1}[0,\infty) \cap (X \times [0,\infty))$$

Let χ be the characteristic function of A_f. Then $\chi \ge 0$, so that Fubini 2 applies and gives

$$(\mu \otimes \mu_L)(A_f) = \int_{X \times \mathbf{R}} \chi \cdot (\mu \otimes \mu_L) = \int_X \left(\int_{\mathbf{R}} \chi(x,t) \cdot \mu_L(t) \right) \cdot \mu(x)$$

But for $x \in X$ fixed, the function $t \in \mathbf{R} \mapsto \chi(x,t)$ is the characteristic function of the interval $[0, f(x)]$. Therefore

$$\int_{\mathbf{R}} \chi(x,t) \cdot \mu_L = \mu_L([0, f(x)]) = f(x)$$

and the identity follows.

□