Math 550. 2nd Midterm. (Some) Solutions.

Problem 1. Let $F : S^1 \to S^1$ be a mapping of the unit circle S^1 into itself.

(a) Prove that if $F(P) = F(-P)$ for all P, then the degree of F is even.

(b) If $F(P) \neq F(-P)$ for all P, then prove that the degree of F is not zero, and that F is surjective.

Solution. This is similar to Problem 5, Homework 6. □

Problem 2. Let X be a topological space.

(a) Define the concepts: (i) X has the *fixed point property*, and(ii) $Y \subset X$ is a *retract* of X.

(b) Prove that if X has the fixed point property and if $Y \subset X$ is a retract of X, then Y also has the fixed point property.

Solution. A subspace $Y \subset X$ is a retract of X if there is a continuous map $r : X \to Y$ such that $r(P) = P$ for all P in Y.

Suppose that X has the fixed point property, that $Y \subset X$ is a retract of X with retraction map $r : X \to Y$ as above, and that $f : Y \to Y$ is continuous. If $i : Y \to X$ the inclusion mapping (given by $i(P) = P$ for all P in Y), then the composite $i \circ f \circ r : X \to X$ is continuous and thus it has a fixed point P in X.

We claim that this point P is in Y and that $f(P) = P$. Indeed, the point $f(r(P))$ is in Y because $r(P)$ is in Y and f maps Y into Y. It follows that P is in Y because $P = i(f(r(P))) = f(r(P))$. If P is in Y, then $r(P) = P$, and so $P = f(P)$. □

Problem 3. Let U and V be two open subsets of the plane.

(a) Prove that if $U \cap V$ is connected and if $H^1U = 0$ and $H^1V = 0$, then $H^1(U \cup V) = 0$.

(b) Prove that if U and V are connected, and $H^1(U \cup V) = 0$, then $U \cap V$ is connected.

Solution. (a) We have two show that every closed one form ω on $U \cup V$ is exact. Since $H^1U = H^1V = 0$, the restrictions of ω to U and to V are both exact. Since $U \cap V$ is connected, Lemma 1.14 in the textbook implies that ω is also exact on $U \cup V$.

(b) To prove that $U \cap V$ is connected is equivalent to proving that the vector space $H^0(U \cap V)$ has dimension 1. Consider the coboundary map

$$\delta : H^0(U \cap V) \to H^1(U \cup V).$$

The kernel of δ equals $H^0(U \cap V)$ because $H^1(U \cup V) = 0$ (by hypothesis). Proposition 5.7 in the textbook says that a locally constant function f on $U \cap V$ is in the kernel of δ if and only if there are locally constant functions f_U on U and f_V on V such that $f = f_U - f_V$ on $U \cap V$. But f_U and f_V must be both constant because, by hypothesis, U and V are both connected. Therefore $f = f_U - f_V$ is the difference of two constant functions and is therefore constant. That is, the kernel of δ consists of the constant functions on $U \cap V$, and thus it has dimension 1. □

Problem 4. Let U be an open subset of the plane.

(a) Define the concepts: (i) 1-chain on U, (ii) 1-cycle on U, and (iii) 1-boundary on U.

(b) Prove that if U is convex, then every 1-cycle on U is a 1-boundary.

Problem 5. Let X be a subset of the plane homeomorphic to the figure 8.

(a) Prove that $\mathbf{R}^2 \setminus X$ has three connected components.

(b) Prove that two of those components are bounded and the other is unbounded.

Solution. The set X is the union $X = A \cup B$, where A and B are closed subsets homeomorphic to a circle with $A \cap B = \{P\}$. Then $(\mathbf{R}^2 \setminus A) \cap (\mathbf{R}^2 \setminus B) = \mathbf{R}^2 \setminus X$ and $(\mathbf{R}^2 \setminus A) \cup (\mathbf{R}^2 \setminus B) = \mathbf{R}^2 \setminus \{P\}$. Consider the coboundary map

$$\delta : H^0((\mathbf{R}^2 \setminus A) \cap (\mathbf{R}^2 \setminus B)) = H^0(\mathbf{R}^2 \setminus X) \longrightarrow H^1((\mathbf{R}^2 \setminus A) \cup (\mathbf{R}^2 \setminus B)) = H^1(\mathbf{R}^2 \setminus \{P\}).$$

A class $[\omega]$ in $H^1(\mathbf{R}^2 \setminus \{P\})$ can be represented as $[\omega] = [\lambda\omega_P]$, for some scalar λ, because the vector space $H^1(\mathbf{R}^2 \setminus \{P\})$ has dimension 1 with basis $[\omega_P]$. The image of δ consists of the classes $[\omega]$ of closed 1-forms on ω on $\mathbf{R}^2 \setminus \{P\}$ such that ω is exact on $\mathbf{R}^2 \setminus A$ and on $\mathbf{R}^2 \setminus B$. Since A is a bounded, connected closed set and P is a point in A, the class $[\omega_P]$ is not zero in $H^1(\mathbf{R}^2 \setminus A)$ (cf. Problem 5, Homework 7). Therefore, the 1-form ω_P is not exact on $\mathbf{R}^2 \setminus A$. Similarly, the 1-form ω_P is not exact on $\mathbf{R}^2 \setminus B$. Therefore, $[\omega] = [\lambda\omega_P]$ is in the image of δ if and only if $\lambda = 0$. That is, the image of δ is the trivial subspace of $H^1(\mathbf{R}^2 \setminus \{P\})$, and so the kernel of δ equals the vector space $H^0(\mathbf{R}^2 \setminus X)$.

There are several methods of showing that the kernel of δ has dimension 3. I will explain two such methods. One is direct and the other is more algebraic. (You can probably simplify the explanation below, but I preferred to spell out all the details.)

Method 1. Because of the Jordan curve theorem we know that the open set $\mathbf{R}^2 \setminus A$ has two connected components, one bounded, say U_0, and the other unbounded, say U_∞, and A is the common boundary of both. Similarly, let V_0 and V_∞ be the bounded and unbounded components, respectively, of $\mathbf{R}^2 \setminus B$. Then

$$\begin{aligned}\mathbf{R}^2 \setminus X = (\mathbf{R}^2 \setminus A) \cap (\mathbf{R}^2 \setminus B) &= (U_0 \cup U_\infty) \cap (V_0 \cup V_\infty) \\ &= (U_0 \cap V_0) \cup (U_0 \cap V_\infty) \cup (U_\infty \cap V_0) \cup (U_\infty \cap V_\infty),\end{aligned}$$

a disjoint union of four open sets.

If f_A is a locally constant function on $\mathbf{R}^2 \setminus A$, then we can represent f as a pair of numbers $f_A = (u_0, u_\infty)$, where u_0 is the (constant) value of f_A on the component U_0 and u_∞ is the value of f_A on U_∞. A locally constant function f_B on $\mathbf{R}^2 \setminus B$ is similarly represented by a pair of numbers (v_0, v_∞). If P is in $\mathbf{R}^2 \setminus X$, then P is exactly in one of the 4 sets $(U_0 \cap V_0)$, $(U_0 \cap V_\infty)$, $(U_\infty \cap V_0)$, or $(U_\infty \cap V_\infty)$, and therefore

$$f(P) = f_A(P) - f_B(P) = \begin{cases} u_0 - v_0, & \text{if } P \text{ is in } U_0 \cap V_0\ ; \\ u_0 - v_\infty, & \text{if } P \text{ is in } U_0 \cap V_\infty; \\ u_\infty - v_0, & \text{if } P \text{ is in } U_\infty \cap V_0; \\ u_\infty - v_\infty, & \text{if } P \text{ is in } U_\infty \cap V_\infty. \end{cases}$$

Thus it seems that we need 4 numbers, $u_0, u_\infty, v_0, v_\infty$, to determine the locally constant function $f = f_A - f_B$ on $(\mathbf{R}^2 \setminus A) \cap (\mathbf{R}^2 \setminus B)$. However, one of these numbers is redundant. Indeed, I claim that f can also be represented as $f = g_A - g_B$, where g_A and g_B are locally constant on $\mathbf{R}^2 \setminus A$ and $\mathbf{R}^2 \setminus B$, respectively, and $g_A = (x, y)$, $g_B = (0, z)$, which will show that only three variables are required to determine f. To see this, suppose that f is given as $f = f_A - f_B$ with $f_A = (u_0, u_\infty)$ and $f_B = (v_0, v_\infty)$. Let $x = u_0 - v_0$, $y = u_\infty - v_0$ and $z = u_\infty - v_\infty$, let g_A be the locally constant function on $\mathbf{R}^2 \setminus A$ given by the pair of numbers $g_A = (x, y)$ (again, this means that $g_A(P) = x = u_0 - v_0$ if P

is in U_0 and $g_A(P) = y = u_\infty - v_0$ if P is in U_∞) and let g_B be given by $g_B = (0, z)$ (and this means that $g_B(P) = 0$ if P is in V_0 and $g_B(P) = z = u_\infty - v_\infty$). If P is in $\mathbf{R}^2 \setminus X$, then we compute the values $g_A(P) - g_B(P)$:

$$g_A(P) - g_B(P) = \begin{cases} x - 0 = u_0 - v_0, & \text{if } P \text{ is in } U_0 \cap V_0\ ; \\ x - z = u_0 - v_\infty, & \text{if } P \text{ is in } U_0 \cap V_\infty; \\ y - 0 = u_\infty - v_0, & \text{if } P \text{ is in } U_\infty \cap V_0; \\ y - z = (u_\infty - v_0) - (v_\infty - v_0) = u_\infty - v_\infty, & \text{if } P \text{ is in } U_\infty \cap V_\infty. \end{cases}$$

By comparing this expression with the expression for f previously displayed we see that $f = g_A - g_B$.

You cannot do with less than 3 numbers because $\mathbf{R}^2 \setminus X$ has at least three components. This is because $\mathbf{R}^2 \setminus X$ is the disjoint union of the four sets $U_0 \cap V_0$, $U_\infty \cap V_0$, $U_0 \cap V_\infty$ and $U_\infty \cap V_\infty$, and at most one of these sets can be empty. Indeed, $U_\infty \cap V_\infty$ is never empty because, as X is compact, there is a disk D that contains X and therefore $U_\infty \cap V_\infty$ contains the complement of D. If one of the other three intersections is empty, then the other two are not. For example, if $U_0 \cap V_0 = \emptyset$, then U_0 must be contained in V_∞ and V_0 must be contained in U_∞. If $U_0 \cap V_\infty = \emptyset$, then U_0 must be contained in V_0, and properly so because $A \cap B = \{P\}$; thus $V_0 \cap U_\infty \neq \emptyset$.

Method 2. The kernel of δ consists of the locally constants functions f on $\mathbf{R}^2 \setminus X$ such that $f = f_A - f_B$, where f_A is locally constant on $\mathbf{R}^2 \setminus A$ and f_B is locally constant on $\mathbf{R}^2 \setminus B$. Consider the map

$$\varphi : H^0(\mathbf{R}^2 \setminus A) \times H^0(\mathbf{R}^2 \setminus B) \longrightarrow H^0((\mathbf{R}^2 \setminus A) \cap (\mathbf{R}^2 \setminus B)) = H^0(\mathbf{R}^2 \setminus X)$$

defined by $\varphi(f, g) = f - g$, for f locally constant on $\mathbf{R}^2 \setminus A$ and g locally constant on $\mathbf{R}^2 \setminus B$. It is apparent that φ is linear at that its image is precisely the kernel of δ (which is $H^0(\mathbf{R}^2 \setminus X)$).

I claim that the kernel of φ consists of all the pairs (f, g) such that f is constant on $\mathbf{R}^2 \setminus A$, g is constant on $\mathbf{R}^2 \setminus B$, and $f - g = 0$ on $\mathbf{R}^2 \setminus X$. This claim implies that the kernel of φ has dimension 1. The Jordan curve theorem implies that the vector spaces $H^0(\mathbf{R}^2 \setminus A)$ and $H^0(\mathbf{R}^2 \setminus B)$ have dimension 2 and thus the product $H^0(\mathbf{R}^2 \setminus A) \times H^0(\mathbf{R}^2 \setminus B)$ has dimension 4. Therefore, the rank-nullity theorem applied to φ implies that $1 + \dim H^0(\mathbf{R}^2 \setminus X) = 4$, or that $H^0(\mathbf{R}^2 \setminus X)$ has dimension 3.

To prove the claim, suppose first a pair (f, g) is in the kernel of φ, that is, suppose that f is locally constant on $\mathbf{R}^2 \setminus A$, that g is locally constant on $\mathbf{R}^2 \setminus B$, and that $f - g = 0$ on $(\mathbf{R}^2 \setminus A) \cap (\mathbf{R}^2 \setminus B)$. This implies that the functions f and g can be glued together (because $f = g$ on $(\mathbf{R}^2 \setminus A) \cap (\mathbf{R}^2 \setminus B)$) to obtain a locally constant function on $(\mathbf{R}^2 \setminus A) \cup (\mathbf{R}^2 \setminus B)$ (cf. Problem 1, Homework 2). But $(\mathbf{R}^2 \setminus A) \cup (\mathbf{R}^2 \setminus B) = \mathbf{R}^2 \setminus \{P\}$ is connected, and so this locally constant function on it must be constant. This implies that if a pair (f, g) is such that $\varphi(f, g) = f - g = 0$, then f and g are both constant and $f = g$. Conversely, if f and g are constant and $f = g$, then $\varphi(f, g) = 0$. $\square$