Math 550. Homework 9. Solutions

Problem 1. Suppose that $U = \mathbf{R}^2 \setminus \{P_1, \cdots, P_n\}$ is the complement of n points in the plane. Prove that the mapping that takes a closed 1-chain γ to $\big(W(\gamma, P_1), \cdots, W(\gamma, P_n)\big)$ determines an isomorphism of H_1U with the free abelian group $\mathbf{Z}^n$.

Solution. Consider the map

$$\Phi : Z_1U \longrightarrow \mathbf{Z}^n$$

given by

$$\Phi(\gamma) = \big(W(\gamma, P_1), \cdots, W(\gamma, P_n)).$$

By the very definition of $W(\gamma, P)$ (winding number of a closed 1-chain γ around a point P not in $\operatorname{supp}\gamma$), this map is a homomorphism of groups.

Let $r > 0$ be smaller that the distance between any pair of points P_i, and let γ_i be a circle or radius r and center P_i. If $m_1, m_2, \cdots, m_n$ are integers, then the closed 1-chain $\gamma = \sum_{i=1}^{n} m_i\gamma_i$ satisfies

$$\Phi(\gamma) = \big(m_1, \cdots, m_n\big),$$

that is to say, the homomorphism Φ is onto.

To compute the kernel of Φ we use Theorem 6.11. If γ is a boundary, then $\Phi(\gamma) = 0$. Conversely, if $\Phi(\gamma) = 0$, then γ has the same winding number around any point not in U as the trivial 1-chain, i.e., γ is homologous to 0, or γ is a boundary in U. Thus the kernel of Φ is precisely $B_1(U)$.

By the first isomorphism theorem for groups, $Z_1U/B_1U \cong \mathbf{Z}^n$. □

Problem 2. (i) Prove that a continuous mapping $F : X \to Y$ determines a homomorphism from Z_1X to Z_1Y taking $B_1X \to B_1Y$, and thus it determines a homomorphism of abelian groups $F_* : H_1X \to H_1Y$.

(ii) Prove that if $F : X \to Y$ and $G : Y \to Z$ are continuous, then $(G \circ F)_* = G_* \circ F_*$ as homomorphism from H_1X to H_1Z. In particular, prove that if X and Y are homeomorphic space, then H_1X and H_1Y are isomorphic abelian groups.

Solution. If $n_1P_1 + \cdots + n_kP_k$ is a 0-chain in X, where the P_i's are points in X, define

$$F_0(n_1\gamma_1 + \cdots + n_kP_k) = n_1F(P_1) + \cdots + n_kF(P_k).$$

Since $C_(X)$ is the free abelian group on the set X, this defines a homomorphism of abelian groups $F_0 : C_0(X) \to C_0(Y)$.

If $\gamma = n_1\gamma_1 + \cdots + n_k\gamma_k$, where the γ_i are continuous paths in X, then define

$$F_1(\gamma) = n_1(F \circ \gamma_1) + \cdots + n_k(F \circ \gamma_k).$$

Since the group of 1-chains on X is the free abelian group on the (classes of) non-degenerate paths in X, it follows that F_1 induces a homomorphism of abelian groups $F_1 : C_1(X) \to C_1(Y)$.

Analogously, if $\Gamma = n_1\Gamma_1 + \cdots + n_k\Gamma_k$, where the Γ_i's are squares in X, set

$$F_2(\Gamma) = n_1(F \circ \Gamma_1) + \cdots + n_k(F \circ \Gamma_k)$$

and note that, by reasons similar to the above, this induces a homomorphism of abelian groups $F_2 : C_2(X) \to C_2(Y)$.

Now we show that if γ is a 1-chain in X, then $F_0(\partial\gamma) = \partial F_1(\gamma)$. By linearity of the maps F_j, it suffices to prove this for $\gamma : [0,1] \to X$ a continuous path in X. In this case, $\partial\gamma = \gamma(1) - \gamma(0)$, and

$$F_0(\partial\gamma) = F(\gamma(0)) - F(\gamma(1)) = \partial(F_1\gamma).$$

It follows that if γ is a closed 1-chain in X, that is, if $\partial\gamma = 0$, then $\partial(F_1(\gamma)) = F_0(\partial\gamma) = F_0(0) = 0$, so that $F_1(\gamma)$ is a closed 1-chain in Y. Therefore, F_1 takes Z_1X into Z_1Y.

Next we show that if Γ is a 2-chain in X, then $\partial(F_2(\Gamma)) = F_1(\partial\Gamma)$. As above, it suffices to show that this is the case for $\Gamma : [0,1] \times [0,1] \to X$ is a square in X. Recall that the boundary of such square Γ is $\partial\Gamma = \gamma_1 + \gamma_2 - \gamma_3 - \gamma_4$, where γ_i are the paths in X determines by the restrictions of Γ to the sides of $[0,1] \times [0,1]$. Then $F_1(\partial\Gamma) = F \circ \gamma_1 + F \circ \gamma_2 - F \circ \gamma_3 - F \circ \gamma_4$. The boundary of the rectangle $F_2(\Gamma) = F \circ \Gamma$ is $\sigma_1 + \sigma_2 - \sigma_3 - \sigma_4$, where the σ_i's are the restrictions of the square $F \circ \Gamma$ to the sides of $[0,1] \times [0,1]$. But then it is easy to see that $\sigma_i = F \circ \gamma_i$, and the identity $\partial(F_2(\Gamma)) = F_1(\partial\Gamma)$ follows. Hence, if γ is a 1-boundary in X, that is $\gamma = \partial\Gamma$, then $F_1(\gamma) = F_1(\partial\Gamma) = \partial F_2(\Gamma)$, so that $F_1(\gamma)$ is a 1-boundary in Y. Thus F_1 takes B_1X into B_1Y and so it induces a homomorphism $F_* : H_1X \to H_1Y$.

The fact that $(G \circ F)_* = G_* \circ F_*$ follows immediately form the associativity of composition of mappings.

Finally, if F is a homomorphism $X \to Y$, then there is a homomorphism $G : Y \to X$ such that $G \circ F = \mathrm{id}_X$ and $F \circ G = \mathrm{id}_Y$. Then $G_* \circ F_* = \mathrm{id}_{H_1X}$ and $F_* \circ G_* = \mathrm{id}_{H_1Y}$, which implies that F_* is one-one and onto. □

Problem 3. Find examples of continuous mappings $F : X \to Y$ such that:

(i) F is one-one, but F_* is not one-one.

(ii) F is surjective, but F_* is not surjective.

Solution. (i) Take $X = \mathbf{R}^2 \setminus \{0\}$, $Y = \mathbf{R}^2$, and F the inclusion mapping. Then F is one-one, but $F_* = 0$. (ii) Take $X = Y = \mathbf{R}^2 \setminus \{0\}$ and $F(z) = z^2$. Then F is surjective, but $F_* : \mathbf{Z} \to \mathbf{Z}$ is the homomorphism $F_*(n) = 2n$, which is not surjective. □

Problem 4. Let K be a compact subset of the plane and let $U = \mathbf{R}^2 \setminus K$. Prove that if K is not connected, and P and Q are in different components of K, then the classes $[\omega_P]$ and $[\omega_Q]$ are linearly independent in H^1U. In particular, prove that if K has n connected components, then H^1U is a vector space of dimension n.

Solution. The connected components of K are closed subsets of K, and therefore are compact subsets of the plane. If A and B are different components of K, then, by Lemma 9.1, there are closed 1-chains γ_A and γ_B in the complement of K such that $W(\gamma_A, P) = 1$ for all P in A and $W(\gamma_B, P) = 0$ for all P in B, and similarly for γ_B.

Therefore, if P is in A, then $\int_{\gamma_A} \omega_P = 1$ and $\int_{\gamma_B} \omega_P = 0$, and if Q is in B, then $\int_{\gamma_B} \omega_Q = 1$ and $\int_{\gamma_Q} \omega_Q = 0$. This implies that $[\omega_P]$ and $[\omega_Q]$ are linearly independent in $H^1 U$.

If K has n components $K_1, \cdots, K_n$, apply Lemma 9.1 with $A = K_i$ and $B = K \setminus K_i$ (which is closed, as finite union of closed sets) to obtain closed 1-chains $\gamma_1, \cdots, \gamma_n$ in $\mathbf{R}^2 \setminus K$ such that $W(\gamma_i, P) = 1$ for all P in K_1 and $W(\gamma, P) = 0$ for all P in $K \setminus K_i$.

Consider the map

$$\Psi : \text{Closed 1-forms on } U \longrightarrow \mathbf{R}^n$$

given by

$$\Psi(\omega) = \left(\int_{\gamma_1} \omega, \cdots, \int_{\gamma_n} \omega \right).$$

This map is linear. By an argument similar to that in Problem 1, it is easy to prove that Ψ is surjective. Moreover, if ω is a closed 1-form on U such that $\Psi(\omega) = 0$, then it follows that $\int_\gamma \omega = 0$ for every closed path γ in U, because any closed path γ in U is homologous to a linear combination $m_1\gamma_1 + \cdots + m_n\gamma_n$, and the integral of a closed 1-form along a closed 1-chain only depends on the homology class of the 1-chain. It follows from this that the kernel of Ψ is the space of exact 1-forms on U, and thus that $H^1 U \cong \mathbf{R}^n$. □

Problem 5. Compute the integral $\displaystyle\int_\gamma \omega$, where ω is the 1-form

$$\omega = \sum_{n=1}^{17} \frac{1}{(x-n)^2 + y^2} \left(-y dx + (x-n) dy\right),$$

and $\gamma(t) = (t\cos(t), t\sin(t))$, $0 \leq t \leq 6\pi$.

Solution. The integral is 53π. □