Math 550. Problems for Solution

Problem 1 (i) For an open subset U of the plane, define the concepts 'closed 1-form on U', and 'exact 1-form on U.'

(ii) True or False (prove or give counterexample): every closed 1-form on U such that $\int_\gamma \omega = 0$ for every closed, piecewise smooth path in U is exact on U.

Problem 2 Given a 1-form ω on an open set U, show that the following are equivalent:

(i) ω is closed;

(ii) $\int_{\partial R} \omega = 0$ for all closed rectangles R contained in U; and

(iii) every point has a neighborhood such that $\int_{\partial R} \omega = 0$ for all closed rectangles R contained in the neighborhood.

Problem 3 Let $P_0 = (0,0)$ and $P_1 = (1,0)$, and let $U = \mathbf{R}^2 \setminus \{P_0, P_1\}$.

(i) Compute the homology groups H_0U and H_1U (that is, find a basis for each group).

(ii) If $F : U \to U$ is the mapping defined by (in complex notation) $F(z) = z^3$, what is the induced homomorphism $F_* : H_1U \to H_1U$ in terms of the basis for H_1U that you have found in (i)?

Problem 4 Let U be an open subset of the plane.

(i) Define $\int_\gamma \omega$ for a closed 1-form ω in U and a 1-chain γ in U.

(ii) Prove that if $\omega = df$ for some smooth function f on U and γ is a 1-chain in U with boundary $\partial\gamma = \sum_j m_j P_j$, then $\int_\gamma df = \sum_j m_j f(P_j)$.

Problem 5 (i) Define '$Y \subset X$ is a retract of X.'

(ii) Let T be a triangle in a plane in $\mathbf{R}^3$, P a point in the interior of T, and L a line through P not in the plane containing P. Prove that T is a retract of $\mathbf{R}^3 \setminus L$.

Problem 6 Let U be an open subset of the plane.

(i) Define $\int_\gamma \omega$ for a closed 1-form ω in U and a 1-chain γ in U.

(ii) Prove that if γ and δ are homologous 1-chains in U, then $\int_\gamma \omega = \int_\delta \omega$.

Problem 7 (i) Let X be a subset of the plane homeomorphic to the figure $\ominus$. Prove that $\mathbf{R}^2 \setminus X$ has three connected components, two of them bounded and the other unbounded.

(ii) Let Y be a closed subset of the plane homeomorphic to the letter Y. Prove that $\mathbf{R}^2 \setminus Y$ is connected.

Problem 8 Let G be a graph consisting of 6 vertices P_1, P_2, P_3, Q_1, Q_2, and Q_3, and 9 edges, each edge joining one of the points P_i to one of the points Q_j. Prove that there is no subset of the plane homeomorphic to G.

Problem 9 Suppose that $U = \mathbf{R}^2 \setminus \{P_1, \cdots, P_n\}$ is the complement of n points in the plane. Prove that the mapping that takes a closed 1-chain γ to $\big(W(\gamma, P_1), \cdots, W(\gamma, P_n)\big)$ determines an isomorphism of $H_1 U$ with the free abelian group $\mathbf{Z}^n$.

Problem 10 Let $F : U \to U'$ be a continuous map. If $\gamma = n_1\gamma_1 + \cdots + n_s\gamma_s$ is a 1-chain in U, with γ_i paths, let $F_*(\gamma)$ be the 1-chain in U' defined by

$$F_*\gamma = n_1(F \circ \gamma_1) + \cdots + n_s(F \circ \gamma_s).$$

If $\sum m_j P_j$ is a 0-chain in U, let $F_*(\sum m_j P_j) = \sum m_j F(P_j)$.

(i) Prove that $\gamma \mapsto F_*\gamma$ is a homomorphism from the group of 1-chains in U to the group of 1-chains in U'. Prove that if γ is closed, then $F_*\gamma$ is also closed.

(ii) Prove that $F_*(\partial\gamma) = \partial(F_*\gamma)$. Prove that if γ is a boundary in U, then $F_*\gamma$ is a boundary in U'.

Problem 11 Let $U \subset V$ be open subsets of the plane.

(i) Prove that if $r : V \to U$ is a retraction, then the induced map $r_* : H_1 V \to H_1 U$ is surjective.

(ii) Prove that if U is n-connected, and V is m-connected, and with $m < n$, then there is no retraction from V onto U.

Problem 12 Let $F : U \to U'$ be a continuous map. Prove that if γ and δ are closed 1-chains in U with the same winding number around all points not in U, then $F_*\gamma$ and $F_*\delta$ are closed 1-chains in U' with the same winding number around all points not in U'.

Problem 13 Let K be a compact, non-empty subset of the plane and let $U = \mathbf{R}^2 \setminus K$.

(i) Prove that if K is not connected, then $H^1 U$ is one-dimensional, generated by $[\omega_P]$ for any P in K.

(ii) Prove that if K is not connected, and P and Q are in different components of K, then the classes $[\omega_P]$ and $[\omega_Q]$ are linearly independent in $H^1 U$.

(iii) Prove that if K has n connected components, then $H^1 U$ is a vector space of dimension n.

Problem 14 (i) Prove that if A and B are compact, connected subsets of the plane such that $A \cap B$ is not connected (and so not empty), then $\mathbf{R}^2 \setminus (A \cup B)$ is not connected.

(ii) Prove that if X is a closed subset of the plane homeomorphic to a closed annulus, then $R^2 \setminus X$ is not connected.

Problem 15 Let A be a connected, closed subset of $\mathbf{R}^2$, and let P and Q be points in A.

(i) Prove that $[\omega_P] = [\omega_Q]$ in $H^1(\mathbf{R}^2 \setminus A)$.

(ii) Prove that A is bounded if and only if $[\omega_P] \neq 0$ in $H^1(\mathbf{R}^2 \setminus A)$.

Problem 16 Compute the integral $\displaystyle\int_\gamma \omega$, where ω is the 1-form

$$\begin{aligned}\omega &= \frac{1}{(x-1)^2+y^2}\big(-ydx+(x-1)dy\big) + \frac{2}{x^2+(y-1)^2}\big(-(y-1)dx+xdy\big)\\ &\quad + \frac{3}{(x+1)^2+y^2}\big(-ydx+(x+1)dy\big) + \frac{4}{x^2+(y+1)^2}\big(-(y+1)dx+xdy\big),\end{aligned}$$

and $\gamma : [0,2] \to \mathbf{R}^2$ is the path given by

$$\gamma = \begin{cases} (-1+\cos(2\pi t), \sin(2\pi t)), & 0 \le t \le 1, \\ (1-\cos(2\pi t), \sin(2\pi t)), & 1 \le t \le 2. \end{cases}$$

Problem 17 Prove that $\mathbf{R}^2$ is not homeomorphic to $\mathbf{R}^3$. (Carefully state the theorems that you use in your proof.)

Problem 18 Let $f : S^1 \to S^1$ be a continuous map of the unit circle with center the origin into itself.

(i) Prove that if $\deg f \neq 1$, then there is a point P in S^1 with $f(P) = P$ and there is a point Q in S^1 with $f(Q) = -Q$.

(ii) Prove that if $f(P) = f(-P)$ for all P, then the degree of f is even.

Problem 19 Let f and g be continuous mappings $S^1 \to S^1$, with $f(1) = g(1) = 1$. Define $f * g : S^1 \to S^1$ as follows:

$$f * g(z) = \begin{cases} f(z^2) & \text{if } z = x+iy,\ y \ge 0, \\ g(z^2), & \text{if } z = x+iy,\ y \le 0. \end{cases}$$

Let $f \cdot g : S^1 \to S^1$ be the mapping $f \cdot g(z) = f(z) \cdot g(z)$. These mappings $f * g$ and $f \cdot g$ are well defined and continuous. Let $f \circ g$ denote the composition $f \circ g(z) = f(g(z))$.

(i) Compute each of $\deg(f \cdot g)$, $\deg(f \circ g)$ and $\deg(f * g)$ in terms of $\deg f$ and $\deg g$.

(ii) Determine which of the following maps are homotopic and which are not homotopic:

$$\mathrm{id}, f, g, f\cdot g, f\cdot g^{-1}, f^{-1}\cdot g, f * \mathrm{id}, f\cdot g, f\cdot g^{-1}, g\cdot f, f\circ g, f\circ g^{-1}, g\circ f, g\circ f^{-1}$$

Write down as many explicit homotopies as you can find between these maps.

Problem 20 (i) Define 'winding number of a continuous path γ around a point P not in γ' and 'paths γ and δ are homotopic.'

(ii) Prove that if γ and δ are closed paths in $\mathbf{R}^2 \setminus \{0\}$, and the segment between $\gamma(t)$ and $\delta(t)$ never hits the origin for all t, then γ and δ have the same winding number around 0.