Math 512B. Homework 9. Due 4/37/08

Problem 1. Let z_1, z_2, z_3 be three distinct nonzero complex numbers. Prove that the following are equivalent:

(i) The points z_1, z_2, and z_3 are the vertices of an equilateral triangle.

(ii) The center of mass of z_1, z_2 and z_3 is the origin.

(iii) There is a complex number a such that z_1, z_2 and z_3 are the solutions to the equation $z^3 = a$.

Problem 2. (i) Prove that $e^{z+w} = e^z \cdot e^w$ for all complex numbers by showing that the power series for e^{z+w} is the product of the series for e^z and e^w.

(ii) Prove that every complex number of absolute value 1 can be written in the form e^{iy} for some real number y.

(iii) Prove that $|e^{x+iy}| = e^x$ for all real x and y.

Problem 3. (i) Prove that exp takes on every complex number except 0.

(ii) Prove that exp is not one-one.

(iii) Given $w \neq 0$, prove that $e^z = w$ if and only if $z = x + iy$, where $x = \log|w|$ and y is any argument for w. (We say that y is an argument for w is $w = |w|(\cos y + i \sin y)$.)

(iv) (Not required) Show that there does not exist a continuous function log that is defined for all nonzero complex numbers and satisfies $\exp(\log z) = z$ foe all $z \neq 0$.

This problem says that any nonzero complex number w has (infinitely many) logarithms, that is solutions z to $e^z = w$, but that there is no continuous way of making a unique choice of a logarithm for all of them.

Problem 4. Compute the first three nonzero terms of the power series centered at 0 representing each of the functions below.

(i) $f(z) = \dfrac{1}{1 - z - z^2}$

(ii) $f(x) = \dfrac{\sin^2 z}{z^2}$

(iii) $f(z) = z \cot z$

(iv) (Not required) $f(z) = \dfrac{\sin(z^2)}{z \cos^2 z}$

Problem 5. The function $f(z) = (e^z - 1)/z$ has a power series representation

$$f(z) = 1 + \frac{z}{2!} + \frac{z^2}{3!} + \cdots$$

and since $f(0) \neq 0$, the function $1/f(z) = z/(e^z - 1)$ has a power series representation

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n$$

with nonzero radius of convergence (as you may suspect, its radius of convergence is 2π).

(i) Prove that $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, and $B_n = 0$ if n is odd and $n > 1$.

(ii) By computing the coefficient of z^n in the right side of

$$z = \left(\sum_{n=0}^{\infty} \frac{B_n}{n!} z^n\right) \left(\frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} \cdots\right)$$

prove that

$$B_n = \frac{-1}{n+1} \sum_{k=0}^{n-1} \binom{n+1}{k} B_k$$

for $n \geq 1$.

(iii) The numbers B_n are called Bernoulli numbers, and are those that have shown up in Problem 1 of the second midterm: $B_n = B_n(0)$. The relation in (ii) permits to compute any B_n in terms of the previous ones. Calculate a few: B_4, B_6, B_8.