Math 512B. Homework 8. Solutions

Problem 1.  (i) Find the Fourier series of the following functions (they are \(2\pi\)-periodic, so only the values on \((-\pi, \pi]\) are given).

(a) \(f(x) = 0 \text{ if } -\pi < x \leq 0 \) and \(f(x) = \sin x \text{ if } 0 < x \leq \pi\).

(b) \(g(x) = x^2 \text{ if } -\pi < x \leq \pi\).

(ii) Discuss the convergence of the Fourier series of each of the functions in (i) and (ii).

Solution.  (ia) The Fourier coefficients for \(f\) are (integrate by parts)

\[
b_k = 0 \text{ for } k \geq 1, \quad a_0 = \frac{2}{\pi}, \quad a_1 = 0, \quad a_k = \frac{1 + \cos(k\pi)}{\pi(1 - k^2)} \text{ for } k \geq 0.
\]

The Fourier series for \(f\) is

\[
S_f(x) = \frac{1}{\pi} + \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{1}{1 - (2k)^2} \cos kx.
\]

(ib) The Fourier coefficients for \(g\) are \(a_0 = \frac{2\pi^2}{3}\), and \(a_k = \frac{4(-1)^k}{k^2}, b_k = 0 \text{ for } k \geq 1\). The Fourier series of \(g\) is

\[
S_g(x) = \frac{\pi^2}{3} + 4 \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \cos(kx).
\]

(ii) Both functions \(f\) and \(g\) are \(2\pi\)-periodic, continuous, and piecewise smooth. Therefore, the Fourier series of \(f\), respectively of \(g\), converges uniformly on \(\mathbb{R}\) to \(f\), respectively, to \(g\).

Problem 2. Find the Fourier series of the function \(f(x) = x(\pi - |x|), -\pi < x \leq \pi\), and show that

\[
\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k - 1)^3} = \frac{\pi^3}{32}.
\]

Solution. The Fourier coefficients are \(a_k = 0\) for all \(k\) (\(f\) is odd) and \(b_k = -\frac{2(-2 + 2(-1)^k)}{k^3\pi}\). The Fourier series for \(f\) is

\[
S_f(x) = \frac{8}{\pi} \sum_{k=1}^{\infty} \frac{1}{(2k - 1)^3} \sin(2k - 1)x.
\]

The function \(f\) is continuous at \(x = \pi/2\), thus \(S_N f(\pi/2) \to f(\pi/2) = \frac{\pi^2}{4}\) as \(N \to \infty\), or

\[
\frac{\pi^2}{4} = \frac{8}{\pi} \sum_{k=1}^{\infty} \frac{1}{(2k - 1)^3} \sin kx
\]

because \(\sin(2k - 1)\pi/2 = (-1)^{k+1}\).

Problem 3. Show that
(i) \( x^3 - \pi^2 x = 12 \sum_{k=1}^{\infty} \frac{(-1)^k}{k^3} \sin kx \), and

(ii) \( x^4 - 2\pi^2 x^2 + \frac{7}{15} \pi^4 = 48 \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^4} \cos kx \).

for all \( x \) in \([-\pi, \pi]\). Hint: Start working with the function \( g \) from Problem 1 (ib)

**Solution.** (i) Using Problem 1(ib), the \( 2\pi \)-periodic function \( f \) given by \( f(x) = 3x^2 - \pi^2 \) for \(-\pi < x \leq \pi\) has \( a_0 = 0 \) and Fourier series

\[
S_f(x) = 12 \sum_{k=1}^{\infty} (-1)^k k^2.
\]

Since \( f \) is piecewise continuous, the function \( F(x) = \int_0^x f = x^3 - \pi^2 x \) has Fourier series

\[
SF(x) = 12 \sum_{k=1}^{\infty} \frac{(-1)^k}{k^3} \sin kx
\]

and since \( F(x) \) is also continuous at all \( x \), \( SF = F \) everywhere.

(ii) This was done in class.

**Problem 4.** Determine which of the following functions are continuous, piecewise continuous, or piecewise smooth on \([-\pi, \pi]\).

(i) \( f(x) = \sqrt{|\sin x|} \).

(ii) \( f(x) = \cos x \) if \( x > 0 \), \( f(x) = -\cos x \) if \( x \leq 0 \).

(iii) \( f(x) = \sin x \) if \( x > 0 \), \( f(x) = \sin 2x \) if \( x \leq 0 \).

(iv) \( f(x) = \sqrt[3]{\sin x} \).

(v) \( f(x) = \sqrt[3]{(\sin x)^3} \).

**Solution.** (i) Continuous. Has derivative \( f'(x) = \pm \frac{\cos x}{\sqrt{|\sin x|}} \) for \( x \) not an integer multiple of \( \pi \). It does not have right or left derivatives at \( x = 0 \).

(ii) Piecewise continuous and piecewise smooth. Has derivative at all \( x \neq 0 \), and it has right and left derivatives at \( x = 0 \).

(iii) Continuous and piecewise smooth. It has derivative at all \( x \neq 0 \), and right and left derivatives at \( x = 0 \).

(iv) Continuous.

(v) Continuous and piecewise smooth, with continuous derivative.

**Problem 5.** (i) Prove that the following Fourier series converge uniformly on \( \mathbb{R} \) to continuous functions \( f \) and \( g \), respectively:

(a) \( f(x) = \sum_{k=0}^{\infty} \frac{1}{2k} \cos kx \).

(b) \( g(x) = \sum_{k=1}^{\infty} \frac{1}{2k} \cos 2^k x \).

(ii) How many derivatives can you guarantee each \( f \) and \( g \) to have?

**Solution.** Done in class.