Math 512B. Homework 8. Solutions

Problem 1. (i) Find the Fourier series of the following functions (they are 2π-periodic, so only the values on $(-\pi, \pi]$ are given).

(a) $f(x) = 0$ if $-\pi < x \le 0$ and $f(x) = \sin x$ if $0 < x \le \pi$.

(b) $g(x) = x^2$ if $-\pi < x \le \pi$.

(ii) Discuss the convergence of the Fourier series of each of the functions in (i) and (ii).

Solution. (ia) The Fourier coefficients for f are (integrate by parts) $b_k = 0$ for $k \ge 1$, and $a_0 = 2/\pi$, $a_1 = 0$, and $a_k = \dfrac{1+\cos(k\pi)}{\pi(1-k^2)}$ for $k \ge 0$. The Fourier series for f is

$$Sf(x) = \frac{1}{\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty} \frac{2}{1-(2k)^2} \cos kx$$

(ib) The Fourier coefficients for g are $a_0 = \dfrac{2\pi^2}{3}$, and $a_k = \dfrac{4(-1)^k}{k^2}$, $b_k = 0$ for $k \ge 1$. The Fourier series of g is

$$Sg(x) = \frac{\pi^2}{3} + 4\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \cos(kx).$$

(ii) Both functions f and g are 2π periodic, continuous, and piecewise smooth. Therefore, the Fourier series of f, respectively of g, converges uniformly on $\mathbf{R}$ to f, respectively, to g. □

Problem 2. Find the Fourier series of the function $f(x) = x(\pi - |x|)$, $-\pi < x \le \pi$, and show that

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)^3} = \frac{\pi^3}{32}.$$

Solution. The Fourier coefficients are $a_k = 0$ for all k (f is odd) and $b_k = -\dfrac{2(-2+2(-1)^k)}{k^3\pi}$. The Fourier series for f is

$$Sf(x) = \frac{8}{\pi}\sum_{k=1}^{\infty} \frac{1}{(2k-1)^3} \sin(2k-1)x.$$

The function f is continuous at $x = \pi/2$, thus $S_N f(\pi/2) \to f(\pi/2) = \dfrac{\pi^2}{4}$ as $N \to \infty$, or

$$\frac{\pi^2}{4} = \frac{8}{\pi}\sum_{k=1}^{\infty} \frac{1}{(2k-1)} \sin kx$$

because $\sin(2k-1)\pi/2 = (-1)^{k+1}$. □

Problem 3. Show that

(i) $x^3 - \pi^2 x = 12 \sum_{k=1}^{\infty} \frac{(-1)^k}{k^3} \sin kx$, and

(ii) $x^4 - 2\pi^2 x^2 + \frac{7}{15}\pi^4 = 48 \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^4} \cos kx$.

for all x in $[-\pi, \pi]$. Hint: Start working with the function g from Problem 1 (ib)

Solution. (i) Using Problem 1(ib), the 2π-periodic function f given by $f(x) = 3x^2 - \pi^2$ for $-\pi < x \leq \pi$ has $a_0 = 0$ and Fourier series

$$Sf(x) = 12 \sum_{k=1}^{\infty} (-1)^k k^2.$$

Since f is piecewise continuous, the function $F(x) = \int_0^x f = x^3 - \pi^2 x$ has Fourier series

$$SF(x) = 12 \sum_{k=1}^{\infty} \frac{(-1)^k}{k^3} \sin kx$$

and since $F(x)$ is also continuous at all x, $SF = F$ everywhere.

(ii) This was done in class. □

Problem 4. Determine which of the following functions are continuous, piecewise continuous, or piecewise smooth on $[-\pi, \pi]$.

(i) $f(x) = \sqrt{|\sin x|}$.

(ii) $f(x) = \cos x$ if $x > 0$, $f(x) = -\cos x$ if $x \leq 0$.

(iii) $f(x) = \sin x$ if $x > 0$, $f(x) = \sin 2x$ if $x \leq 0$.

(iv) $f(x) = \sqrt[3]{\sin x}$.

(v) $f(x) = \sqrt[3]{(\sin x)^4}$.

Solution. (i) Continuous. Has derivative $f'(x) = \pm \frac{\cos x}{\sqrt{|\sin x|}}$ for x not an integer multiple of π. It does not have right or left derivatives at $x = 0$.

(ii) Piecewise continuous and piecewise smooth. Has derivative at all $x \neq 0$, and it has right and left derivatives at $x = 0$.

(iii) Continuous and piecewise smooth. It has derivative at all $x =\neq 0$, and right and left derivatives at $x = 0$.

(iv) Continuous.

(v) Continuous and piecewise smooth, with continuous derivative. □

Problem 5. (i) Prove that the following Fourier series converge uniformly on $\mathbf{R}$ to continuous functions f and g, respectively:

(a) $f(x) = \sum_{k=0}^{\infty} \frac{1}{2^k} \cos kx$.

(b) $g(x) = \sum_{k=1}^{\infty} \frac{1}{2^k} \cos 2^k x$.

(ii) How many derivatives can you guarantee each f and g to have?

Solution. Done in class. □