# Math 512B. Homework 8. Due 4/16/08

**Problem 1.** (i) Find the Fourier series of the following functions (they are $2\pi$-periodic, so only the values on $(-\pi, \pi]$ are given).

(a) $f(x) = 0$ if $-\pi < x \leq 0$ and $f(x) = \sin x$ if $0 < x \leq \pi$.

(b) $g(x) = x^2$ if $-\pi < x \leq \pi$.

(ii) Discuss the convergence of the Fourier series of each of the functions in (i) and (ii).

**Problem 2.** Find the Fourier series of the function $f(x) = x(\pi - |x|)$, $-\pi < x \leq \pi$, and show that

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)^3} = \frac{\pi^3}{32}.$$

**Problem 3.** Show that

(i) $x^3 - \pi^2 x = 12 \displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k^3} \sin kx$, and

(ii) $x^4 - 2\pi^2 x^2 + \dfrac{7}{15}\pi^4 = 48 \displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^4} \cos kx$.

for all $x$ in $[-\pi, \pi]$. Hint: Start working with the function $g$ from Problem 1 (ib)

**Problem 4.** Determine which of the following functions are continuous, piecewise continuous, or piecewise smooth on $[-\pi, \pi]$.

(i) $f(x) = \sqrt{|\sin x|}$.

(ii) $f(x) = \cos x$ if $x > 0$, $f(x) = -\cos x$ if $x \leq 0$.

(iii) $f(x) = \sin x$ if $x > 0$, $f(x) = \sin 2x$ if $x \leq 0$.

(iv) $f(x) = \sqrt[3]{\sin x}$.

(v) $f(x) = \sqrt[3]{(\sin x)^4}$.

**Problem 5.** (i) Prove that the following Fourier series converge uniformly on $\mathbf{R}$ to continuous functions $f$ and $g$, respectively:

(a) $f(x) = \displaystyle\sum_{k=0}^{\infty} \frac{1}{2^k} \cos kx$.

(b) $g(x) = \displaystyle\sum_{k=1}^{\infty} \frac{1}{2^k} \cos 2^k x$.

(ii) How many derivatives can you guarantee each $f$ and $g$ to have?