Math 512B. Homework 7. Solutions

Problem 1. For each positive integer N, let

$$F_N(t) = \frac{1}{N+1}\sum_{n=0}^{N} D_n(t)$$

denote the **Fejer kernel**, defined on $[-\pi, \pi]$.

(a) Prove that

$$F_N(t) = \frac{1}{N+1}\frac{\sin^2((N+1)t/2)}{2\sin^2(t/2)}.$$

(b) The function F_N is periodic and non-negative.

(c) The integral $\int_{-\pi}^{\pi} F_N = \pi$.

(d) For each $a > 0$,

$$\lim_{N\to\infty}\int_{a\le|t|\le\pi} F_N(t)\,dt = 0.$$

Solution. (a) The Dirichlet kernel $D_n(t) = \dfrac{1}{2} + \cos t + \cdots + \cos nt = \dfrac{\sin(N+1/2)t}{2\sin t/2}$.

(b) Part (a) shows that $F_N \ge 0$. It is 2π-periodic because it is a sum of 2π-periodic functions.

(c) We have proven in class that $\int_0^{\pi} D_n(t) = \pi/2$, so

$$\int_{-\pi}^{\pi} F_N = 2\int_0^{\pi} F_N = \frac{1}{N+1}\sum_{n=0}^{N}\int_0^{\pi} D_n = \pi.$$

Thus

$$F_N(t) = \frac{1}{2(N+1)\sin(t/2)}\sum_{n=0}^{N}\sin(2N+1)(t/2)$$

and the result follows because for any real number x not a multiple of π, we have

$$\sin x + \sin 3x + \cdots + \sin(2n-1)x = \frac{\sin^2 nx}{\sin x}.$$

(d) If $0 < a \le |t| \le \pi$, then there is a constant $C_a > 0$ such that $\sin^2(t/2) \ge C_a$. Hence $F_N(t) \le \dfrac{1}{C_a(N+1)}$ for $a \le |t| \le \pi$, and so

$$0 \le \int_a^{\pi} F_N \le \frac{\pi - a}{C_a(N+1)} \to 0, \quad \text{as } N\to\infty$$

$\square$

Problem 2. Let f be integrable on $[-\pi, \pi]$. For each positive integer N, define

$$\sigma_N f(x) = \frac{1}{N+1}\sum_{n=0}^{N} S_n f(x).$$

(a) Prove that
$$\sigma_N f(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} F_N(x-t)f(t)\,dt.$$

(b) Show that if f is continuous on $[-\pi,\pi]$ and $f(-\pi)=f(\pi)$, then $\sigma_N f$ converges to f uniformly on $[-\pi,\pi]$ as $N\to\infty$.

Solution. (a) We have proven that $S_n f(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} D_n(x-t)f(t)\,dt$. Therefore (sums below are finite)

$$\begin{aligned}
\sigma_N f(x) &= \frac{1}{N+1}\sum_{n=0}^{N} S_n f(x) \\
&= \frac{1}{N+1}\frac{1}{\pi}\sum_{n=0}^{N} D_n(x-t)f(t)\,dt \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{N+1}\sum_{n=0}^{N} D_n(x-t)f(t)\,dt \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} F_N(x-t)f(t)\,dt.
\end{aligned}$$

□

Problem 3. If f has continuous derivative on $[-\pi,\pi]$ then there is a constant M such that $|a_k|+|b_k|\le M/k$ for all $k>0$.

Solution. The hypothesis imply that the periodic extension of f is piecewise smooth. The Fourier series of f' are $a_k' = kb_k$ and $b_k' = -ka_k$ for $k\ge 1$, and thus, since f' is continuous on $[-\pi,\pi]$, the series

$$\sum_{k=1}^{\infty} k^2(|b_k|^2+|a_k|^2) \le \frac{1}{2\pi}\int_{-\pi}^{\pi}|f'|^2 < \infty$$

converges. Thus the sequence $k^2(|a_k|^2+|b_k|^2)$ is bounded. If $M>0$ is such that $k^2(|a_k|^2+|b_k|^2)\le M$ for all $k\ge 1$, then $|a_k|+|b_k|\le 2\sqrt{M}/k$ for all $k\ge 1$. □

Problem 4. Let f be a continuous function on $[-\pi,\pi]$, and suppose that all its Fourier coefficients $a_k=0$ for $k\ge 0$, and $b_k=0$ for $k\ge 1$. Show that f is identically equal to 0.

Solution. The periodic extension of f to $\mathbf{R}$ is 2π-periodic and piecewise continuous, but it may not be continuous, so its Fourier series may not converge to the value of the function for all x. However, because $(1/\pi)\int_{-\pi}^{\pi} f = a_0 = 0$, the function $F(x)=\int_0^x f$ is 2π-periodic and continuous, with piecewise continuous derivative. It thus equals its Fourier series

$$F(x) = \frac{A_0}{2} + \sum_{k=1}^{\infty}(A_k\cos kx + B_k\sin kx)$$

for all x. As we know, the Fourier coefficients of F and f are related by $A_k=-b_k/k$ and $B_k=a_k/k$ for all $k\ge 1$. Therefore, $A_k=B_k=0$ for $k\ge 1$. It follows that $F(x)=A_0/2$ for all x; thus F is a constant function and its derivative $F'(x)=f(x)=0$ for all x. □

Problem 5. Perhaps you have heard about the Dirac delta function. This is the "function" δ on $[-\pi,\pi]$ such that

$$\int_{-\pi}^{\pi} f(x)\delta(x-a)\,dx = f(a).$$

(i) Although δ is not really a function, you may pretend that it is and try to compute its Fourier series. What is this series?

(ii) Suppose that δ was not only just a function, but also a differentaible function with derivative δ'. Compute the Fourier series of δ'.

Solution. (i) At least formally, the Fourier coefficients of δ are

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \delta(x)\cos kx\,dx = \cos 0 = 1,$$

for all k, and

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \delta(x)\sin kx\,dx = \sin 0 = 0,$$

for all k. The Fourier series for δ is thus

$$\delta(x) \sim \frac{1}{2} + \sum_{k=1}^{\infty} \cos kx.$$

(ii) The Fourier series for the "derivative" δ' is

$$\delta'(x) \sim -\sum_{k=1}^{\infty} k\sin kx.$$

□