Math 512B. Homework 6. Due 3/12/08

(Revised 3/6)

Problem 1. Let m and n be natural numbers. Prove:

(i) $\displaystyle\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = 0$ if $m \neq n$;

(ii) $\displaystyle\int_{-\pi}^{\pi} \cos^2 nx = \begin{cases} \pi & n \neq 0; \\ 2\pi & n = 0; \end{cases}$

(iii) $\displaystyle\int_{-\pi}^{\pi} \sin mx \sin nx = 0$ if $m \neq n$;

(iv) $\displaystyle\int_{-\pi}^{\pi} \sin^2 nx \, dx = \begin{cases} 0 & n = 0; \\ \pi & n \neq 0; \end{cases}$

(v) $\displaystyle\int_{-\pi}^{\pi} \cos mx \sin nx \, dx = 0.$

Problem 2 (Not required)**.** A function s on $[a, b]$ is called a step function if there is a partition $P = \{t_0, t_1, \cdot, t_n\}$ of $[a, b]$ such that s is constant on each interval (t_{i-1}, t_i).

(i) Suppose that s is a step function on $[a, b]$. Prove that s is integrable on $[a, b]$ and that

$$\int_a^b s(x)\, dx = \sum_{i=1}^{n} \sigma_i (t_i - t_{i-1})$$

where $\{t_0, t_1, \cdots, t_n\}$ is the partition for which $s(x) = \sigma_i$ (a constant) for all x in (t_{i-1}, t_i).

(ii) Suppose that f is integrable on $[a, b]$. Prove that for any $\varepsilon > 0$ there is a step function s_u on $[a, b]$ such that $\displaystyle\int_a^b s_u - \int_a^b f < \varepsilon$, and a step function s_l such that $\displaystyle\int_a^b f - \int_a^b s_l < \varepsilon$.

(iii) Suppose that for all ε there are step functions $s_l \leq f \leq s_u$ such that $\displaystyle\int_a^b s_u - \int_a^b s_l < \varepsilon$. Prove that f is integrable on $[a, b]$.

Note. While not required, this is an interesting and easy problem. Incidentally, we did something like this in class last semester when we discussed the Lebesgue integral.

Problem 3. The objective of this problem is to prove the Riemann-Lebesgue lemma, which states that if f is integrable on $[a, b]$, then

$$\lim_{\lambda \to \infty} \int_a^b f(x) \sin \lambda x \, dx = 0.$$

(i) Prove, by computing the integral explicitly, that $\displaystyle\lim_{\lambda \to \infty} \int_c^d \sin \lambda x \, dx = 0.$

(ii) Prove that if s is a step function on $[a, b]$, then $\lim_{\lambda \to \infty} \int_a^b s(x) \sin \lambda x \, dx = 0$.

(iii) Use Problem 2 to prove that if f is integrable on $[a, b]$, then $\lim_{\lambda \to \infty} \int_a^b f(x) \sin \lambda x \, dx$.

Problem 4. For each function f below, compute its Fourier series and determine if that Fourier series at $x = 0$ converges to $f(0)$.

(i) $f(x) = |x|$ for x in $[-\pi, \pi]$.

(ii) $f(x) = \sin x + \cos 2x$, for x in $[-\pi, \pi]$.

(iii) $f(x) = \exp x$, for x in $[-\pi, \pi]$.

Problem 5. A function f defined on $[-\pi, \pi]$ is an even function if $f(-x) = f(x)$ for all x, and is an odd function if $f(-x) = -f(x)$ for all x.

(i) Prove that if f is even, then $\int_{-\pi}^{\pi} f = 2 \int_0^{\pi} f$.

(ii) Prove that if f is odd, then $\int_{-\pi}^{\pi} f = 0$.

(iii) Prove that for any function f defined on $[-\pi, \pi]$, then function g on $[-\pi, \pi]$ given by $g(x) = f(x) + f(-x)$ is even, and the function h given by $h(x) = f(x) - f(-x)$ is odd.

(iv) Prove that if a function f (integrable on $[-\pi, \pi]$) is even, then its Fourier coefficients $b_1 = b_2 = \cdots = 0$, while if f is odd, then $a_0 = a_1 = \cdots = 0$.

Problem 6 (Not required)**.** Let f be an integrable function on $[-1, 1]$.

(i) Prove that the function

$$A(t) = \int_{-\pi}^{\pi} (f - t \cos nx)^2 \, dx$$

reaches a minimun when

$$t = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx.$$

(ii) Prove that the function

$$B(t) = \int_{-\pi}^{\pi} (f - t \sin nx)^2 \, dx$$

reaches a minimun when

$$t = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx.$$

Note. Initially I had in mind to assign this problem, but I forgot!