Math 512B. Homework 5. Solutions

Problem 1. Determine whether each of the following series converges or does not converge.

(i) \( \sum_{n=1}^{\infty} \frac{\cos n\alpha}{n^2} \).

Solution. (i) Converges by the comparison test: \( \left| \frac{\cos n\alpha}{n^2} \right| \leq \frac{1}{n^2} \).

(ii) Converges by Leibniz’s Theorem: it is an alternating series and \( \log n/n \geq \log(n+1)/(n+1) \) for \( n \geq 3 \).

(iii) Diverges by the comparison test: \( \frac{1}{n} \leq \frac{\log n}{n} \) for \( n > 3 \).

(iv) Converges by the comparison test: \( 0 \leq \frac{\sqrt{n+1} - \sqrt{n}}{n} = \frac{1}{n(\sqrt{n+1} + \sqrt{n})} \leq \frac{1}{2n^{3/2}} \).

(v) Converges by the comparison test: \( \frac{1}{(\log n)^n} \leq \frac{1}{2^n} \) for \( n \geq 9 \).

Problem 2. (i) Suppose that \( f \) is nondecreasing on \([1, \infty)\). Prove that

\[
\int_{1}^{n} f(1)+f(2)+\cdots+f(n-1) \leq \int_{1}^{n} f(2)+f(3)+\cdots+f(n).
\]

(ii) Take \( f = \log \) in (i) and prove that

\[
e^{1-n} n^2 \leq n! \leq e^{n-1} (n+1)^{n+1}
\]

(iii) Prove that

\[
\lim_{n \to \infty} \frac{\sqrt[n]{n!}}{n} = e^{-1}.
\]

(iv) Prove that the series \( \sum_{n=1}^{\infty} \frac{a^n n!}{n^n} \) converges if \( a < e \).

(v) Prove that \( \sum_{n=1}^{\infty} \frac{a^n n!}{n^n} \) does not converge if \( a \geq e \).

Solution. (i) If \( f \) is nondecreasing, then \( f(k) \leq \int_{k}^{k+1} f \leq f(k+1) \) for all \( k \geq 1 \).

(ii) The integral \( \int_{1}^{n} \log n \log n - n \). Because of the properties of log, \( \log k! = \log k + \log(k-1) + \cdots + \log 1 \), so (i) implies that \( \log(n-1)! \leq \log n^n - n \leq \log n! \), since exp is increasing, \( (n-1)! \leq n^n e^{-n} \leq n! \), which is equivalent to the stated inequality.

(iii) Obvious by virtue of (ii).

(iv) and (v) follow from the ratio test because \( \frac{a^{n+1}(n+1)!/(n+1)^{n+1}}{a^n n!/n^n} = a \left( \frac{n}{n+1} \right)^n \) converges to \( a \) as \( n \to \infty \).

Problem 3. (i) Let \( a_n \geq 0 \). Prove that if \( \sum_{n=1}^{\infty} a_n \) does not converge, then \( \sum_{n=1}^{\infty} \frac{a_n}{1+a_n} \) does not converge.

(ii) Let \( a_n \geq 0 \). Prove that if \( b_n \) is a bounded sequence and \( \sum_{n=1}^{\infty} a_n \) converges, then \( \sum_{n=1}^{\infty} a_n b_n \) also converges.

(iii) Let \( a_n \geq 0 \). Prove that if \( \lim_{n \to \infty} \sqrt[n]{a_n} < 1 \), then \( \sum_{n=1}^{\infty} a_n \) converges; and if \( \lim_{n \to \infty} \sqrt[n]{a_n} > 1 \), then \( \sum_{n=1}^{\infty} a_n \) does not converge.

(iv) (Not required) Let \( a_n \geq 0 \). Prove that if \( a_n \) is decreasing and \( \sum_{n=1}^{\infty} a_n \) converges, then \( \lim_{n \to \infty} na_n = 0 \).

(v) Suppose that \( \sum_{n=1}^{\infty} a_n \) converges. Let \( n_1 < n_2 < n_3 < \cdots \) be an increasing sequence of natural numbers. From the sequence \( a_n \) obtain a new sequence \( b_n \) by setting

\[
b_1 = a_1 + a_2 + \cdots + a_{n_1},
b_2 = a_{n_1+1} + a_{n_1+2} + \cdots + a_{n_2},
\]

\[
\vdots
\]

\[
b_k = a_{n_{k-1}+1} + a_{n_{k-1}+2} + \cdots + a_{n_k}.
\]

Prove that \( \sum_{n=1}^{\infty} b_n \) also converges and \( \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n \).

Solution. (i) If \( \sum_{n=1}^{\infty} \frac{a_n}{1+a_n} \) converges, then the sequence \( \frac{a_n}{1+a_n} \) converges to 0 as \( n \to \infty \). Because of this, there is \( N \) such that \( a_n \leq 1 \) for \( n > N \), and so \( a_n \leq 2 \frac{a_n}{1+a_n} \) for all \( n > N \). By the comparison test, the series \( \sum_{n=N+1}^{\infty} a_n \) converges and thus the original series \( \sum_{n=1}^{\infty} a_n \) converges.
(ii) If $|b_n| \leq M$, then $|a_nb_n| \leq Ma_n$, so $\sum_{n=1}^{\infty} |a_nb_n|$ converges by the comparison test.

**Problem 4.** Let $b_n \neq 0$. We say that the infinite product $\prod_{n=1}^{\infty} b_n$ converges if the sequence of partial products $p_n = \prod_{k=1}^{n} b_k$ converges and $\lim_{n \to \infty} p_n \neq 0$.

(i) Prove that if $\prod_{n=1}^{\infty} b_n$ converges, then $\lim_{n \to \infty} b_n = 1$.

(ii) Suppose that $b_n > 0$. Prove that $\prod_{n=1}^{\infty} b_n$ converges if and only if $\sum_{n=1}^{\infty} \log b_n$ converges.

(iii) (Not required) Suppose that $a_n \geq 0$. Prove that $\prod_{n=1}^{\infty} (1 + a_n)$ converges if and only if $\sum_{n=1}^{\infty} a_n$ converges.

(iv) (Not required) Prove that $\prod_{n=2}^{\infty} \left( 1 - \frac{1}{n^\alpha} \right)$ converges if and only if $\alpha > 1$.

(v) (Not required) Evaluate the infinite product $\prod_{n=2}^{\infty} \left( 1 - \frac{1}{n^2} \right)$.

**Solution.**
(i) When studying the Taylor remainder for $e^x$, we have shown that $\left| e - 1 - \sum_{n=1}^{N} \frac{1}{n!} \right| \leq \frac{1}{(N+1)!}$, so the partial sums $s_N = \sum_{n=1}^{N} \frac{1}{n!}$ converge to $e - 1$.

(ii) Argue as in (i) using the Taylor series for $\log(1-x)$ at $x = 1/2$.

(iii) (There was a typo.) Use the Taylor series for $\cos x$ at $\pi$.

Problem 5. Prove that each of the following series converge to the given limit.

(i) $\sum_{n=1}^{\infty} \frac{1}{n!} = e - 1$

(ii) $\sum_{n=1}^{\infty} \frac{1}{n2^n} = \log 2$

(iii) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\pi^2n}{(2n)!} = 2$

(iv) (Not required) $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

(v) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{2n+1}{n^2+n} = 1$

(vi) (Not required) $\sum_{n=1}^{\infty} (-1)^n \frac{(2n)!}{2^{4n}(n!)^2} = \frac{2}{\sqrt{5}} - 1$. 

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