Math 512B. Homework 5. Due 2/27/08

(Revised 2/27)

Problem 1. Determine whether each of the following series converges or does not converge.

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n\alpha}{n^2}$.

(ii) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\log n}{n}$

(iii) $\displaystyle\sum_{n=1}^{\infty} \frac{\log n}{n}$.

(iv) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{n}$

(v) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(\log n)^n}$.

Problem 2. (i) Suppose that f is nondecreasing on $[1,\infty)$. Prove that

$$f(1)+f(2)+\cdots+f(n-1) \le \int_1^n f \le f(2)+f(3)+\cdots+f(n).$$

(ii) Take $f=\log$ in (i) and prove that

$$e^{1-n}n^n \le n! \le e^{-n}(n+1)^{n+1}$$

(iii) Prove that

$$\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \frac{1}{e}.$$

(iv) Prove that the series $\displaystyle\sum_{n=1}^{\infty} \frac{a^n n!}{n^n}$ converges if $a<e$.

(v) Prove that $\displaystyle\sum_{n=1}^{\infty} \frac{a^n n!}{n^n}$ does not converge if $a \ge e$.

Problem 3. (i) Let $a_n \ge 0$. Prove that if $\displaystyle\sum_{n=1}^{\infty} a_n$ does not converge, then $\displaystyle\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$ does not converge.

(ii) Let $a_n \ge 0$. Prove that if b_n is a bounded sequence and $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, then $\displaystyle\sum_{n=1}^{\infty} a_n b_n$ also converges.

(iii) Let $a_n \ge 0$. Prove that if $\displaystyle\limsup_{n\to\infty} \sqrt[n]{a_n} < 1$, then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges; and if $\displaystyle\limsup_{n\to\infty} \sqrt[n]{a_n} > 1$, then $\displaystyle\sum_{n=1}^{\infty} a_n$ does not converge.

(iv) (Not required) Let $a_n \geq 0$. Prove that if a_n is decreasing and $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} na_n = 0$.

(v) Suppose that $\sum_{n=1}^{\infty} a_n$ converges. Let $n_1 < n_2 < n_3 < \cdots$ be an increasing sequence of natural numbers. From the sequence a_n obtain a new sequence b_n by setting

$$\begin{aligned} b_1 &= a_1 + a_2 + \cdots + a_{n_1} \\ b_2 &= a_{n_1+1} + a_{n_1+2} + \cdots + a_{n_2} \\ &\vdots \\ b_k &= a_{n_{k-1}+1} + a_{n_{k-1}+2} + \cdots + a_{n_k} \end{aligned}$$

Prove that $\sum_{n=1}^{\infty} b_n$ also converges and $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n$

Problem 4. Let $b_n \neq 0$. We say that the infinite product $\prod_{n=1}^{\infty} b_n$ converges if the sequence of partial products $p_n = \prod_{k=1}^{n} b_k$ converges and $\lim_{n\to\infty} p_n \neq 0$.

(i) Prove that if $\prod_{n=1}^{\infty} b_n$ converges, then $\lim_{n\to\infty} b_n = 1$.

(ii) Suppose that $b_n > 0$. Prove that $\prod_{n=1}^{\infty} b_n$ converges if and only if $\sum_{n=1}^{\infty} \log b_n$ converges.

(iii) (Not required) Suppose that $a_n \geq 0$. Prove that $\prod_{n=1}^{\infty} (1 + a_n)$ converges if and only if $\sum_{n=1}^{\infty} a_n$ converges.

(iv) (Not required) Prove that $\prod_{n=2}^{\infty} \left(1 - \frac{1}{n^\alpha}\right)$ converges if and only if $\alpha > 1$.

(v) (Not required) Evaluate the infinite product $\prod_{n=2}^{\infty} \left(1 - \frac{1}{n^2}\right)$

Problem 5. Prove that each of the following series converge to the given limit.

(i) $\sum_{n=1}^{\infty} \frac{1}{n!} = e - 1$

(ii) $\sum_{n=1}^{\infty} \frac{1}{n2^n} = \log 2$

(iii) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\pi^{2n}}{(2n)!} = 2$

(iv) (Not required) $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

(v) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{2n+1}{n^2+n} = 1$

(vi) (Not required) $\sum_{n=1}^{\infty} (-1)^n \frac{(2n)!}{2^{4n}(n!)^2} = \frac{2}{\sqrt{5}} - 1$.