Math 512B. Homework 4. Due 2/20/08

(Revised 2/18)

Problem 1. (i) Prove that the remainder $R_{2n+1,0,\arctan}(x)$ of degree $2n+1$ of the function arctan satisfies

$$|R_{2n+1,0,\arctan}(x)| \le \frac{1}{2n+3}$$

for $|x| \le 1$.

(ii) Use the relation $\arctan x + \arctan y = \arctan\left(\frac{x+y}{1-xy}\right)$ (valid for $xy \ne 1$) to show that

$$\frac{\pi}{4} = \arctan\frac{1}{2} + \arctan\frac{1}{3},$$

and that

$$\frac{\pi}{4} = 4\arctan\frac{1}{5} - \arctan\frac{1}{239}.$$

(This last identity and its use for estimating π are due to J. Machin.)

(iii) Use the above and the appropriate Taylor polynomial for arctan to show that $\pi = 3.14159\cdots$. That is, compute the first 5 decimal digits of π exactly. (Be careful with many little errors, they tend to add up to a big one.)

Problem 2. For any real number α and integer $n \ge 0$, define the binomial coefficient $\binom{\alpha}{n}$ by

$$\binom{\alpha}{n} = \begin{cases} 1, & n = 0 \\ \dfrac{\alpha\cdot(\alpha-1)\cdot\ldots\cdot(\alpha-n+1)}{1\cdot 2\cdot\ldots\cdot n} & n > 0. \end{cases}$$

(i) (Not required) Let $f(x) = (1+x)^m$, where m is a non-negative integer. What is the Taylor polynomial of degree n for f at 0?

(ii) Let $f(x) = (1+x)^\alpha$, where α is any real number, not necessarily a natural number. Prove that the Taylor polynomial of degree n for f at 0 is

$$P_{n,0,f}(x) = \sum_{k=0} \binom{\alpha}{k} x^k.$$

(iii) Find the integral form of the remainder $R_{n,0,f}(x)$ of degree n for $f(x) = (1+x)^\alpha$ at 0, and prove that if $|x| < 1$, then

$$\lim_{n\to\infty} R_{n,0,f}(x) = 0.$$

The Taylor polynomials of $f(x) = (1+x)^\alpha$ can be used to approximate radicals quite fast and accurately. For example, if your want to find the first few decimals of $\sqrt[q]{p}$ (where p is a positive number). However if you simply write $p^{1/q} = (1+x)^\alpha$ with $x = p-1$ and $\alpha = 1/q$, you may have $x > 1$ and in order to obtain a good approximation to $\sqrt[q]{p}$ you may be required to compute the remainder of very high degree for f at 0.

(iii) To experience what I mean above, try it with $\sqrt{11} = (1+10)^{1/2}$. How large must n be so that the remainder $R_{n,0}(10) < 1/10^2$?

There is a trick for approximating $\sqrt[q]{p}$ using Taylor polynomials which will speed up your calculations. For that you pick a number d for which $\sqrt[q]{d}$ is known and such that $0 < p/d < 2$. Then write

$$\frac{p}{d} = 1 + x$$

with $|x| < 1$ and obtain

$$p^{1/q} = d^{1/q}(1+x)^{1/q}.$$

(iv) Use the above for approximating $\sqrt{11}$ with an error $< 1/10^7$. (There are many choices for d, choose wisely!)

Problem 3 (Based on Giesy, *Mathematics Magazine*, 45 (1972), pp 148–149.)**.** Let f_n be the function defined by

$$f_n(x) = \frac{1}{2} + \cos 2x + \cos 4x + \cdots + \cos 2nx$$

(i) Show that

$$f_n(x) = \frac{\sin[(2n+1)x]}{2\sin x}$$

if x is not an integral multiple of π. (Hint. Use the trigonometric identity $\sin(\alpha+\beta) - \sin(\alpha-\beta) = 2\cos\alpha\sin\beta$.)

(ii) Let E_n be the number defined by

$$E_n = \int_0^{\pi/2} 2x f_n(x)\, dx$$

Prove that

$$E_n = \frac{\pi^2}{8} + \sum_{k=1}^{n} \frac{(-1)^k - 1}{2k^2}$$

In particular, for odd indexes:

$$\frac{\pi^2}{8} = E_{2n-1} + \sum_{k=1}^{n} \frac{1}{(2k-1)^2}.$$

(iii) Prove that

$$E_{2n-1} = \frac{1}{4n-1}\left[1 + \int_0^{\pi/2} u'(x)\cos(4n-1)x\, dx\right]$$

Hint. Let $u(x) = x/\sin x$ if $0 < x \le \pi/2$ and $u(0) = 0$, and let $v(x) = \sin(4n-1)x$; show that u' and v' are continuous on $[0, \pi/2]$, and the apply integration by parts to the integral used to define E_{2n-1} in (ii).

(iv) Prove that $0 \le u(x) \le \pi/2$ for x in $[0, \pi/2]$.

(v) Prove that $\lim_{n\to\infty} E_{2n-1} = 0$.

Problem 4. Let $f(x) = \log(1+x)$.

(i) Find the integral expression for the remainder $R_{n,0}(x)$ of degree n for f at 0.

(ii) Prove that if $0 < x \le 1$, then

$$|R_{n,0}(x)| < \frac{1}{n+1}.$$

(iii) Prove that if $-1 < x \le 0$, then

$$|R_{n,0}(x)| < \frac{1}{(1+x)(n+1)}.$$