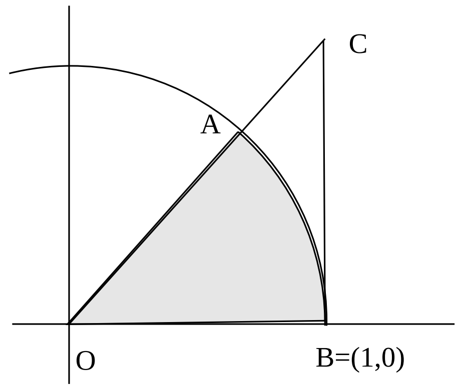
Math 512B. Homework 2. Solutions

Problem 1. The sector in the figure has area $\frac{x}{2}$.



(i) By considering the area of the triangles OAB and OCB prove that if $0 < x < \frac{\pi}{4}$, then

$$\frac{\sin x}{2} < \frac{x}{2} < \frac{\sin x}{2\cos x}.$$

(ii) Prove that, if $|x| < \frac{\pi}{4}$, then

$$\cos x < \frac{\sin x}{x} < 1.$$

(iii) Prove that

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

(iv) Find the limit

$$\lim_{x\to 0} \frac{1-\cos x}{x}.$$

(v) Find $\sin'$ starting from the definition of the derivative. (Use (a)–(d) above, and the addition formula for sin.)

Solution.(i) The height of triangle OAB is $\sin x$, so its area is $\sin x/2$. The height of OBC is $\sin x/\cos x$, so its area is $\sin x/2\cos x$.

(ii) Immediate from (i).

(iii) It follows from the definition of $\cos x$ that $\lim_{x\to 0} \cos x = 1$. Therefore, also $\lim \sin x/x = 1$ by using the inequalities in (ii)

(iv) Multiply and divide by $1 + \cos x$, use the identity $(\cos x)^2 + (\sin x)^2 = 1$, and parts (ii) and (iii) to obtain that $\lim_{x\to 0} \frac{1-\cos x}{x} = 0$.

(v) By definition, if $0 < x < \pi/4$,

$$\begin{aligned}
\sin' x &= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} \\
&= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\
&= \lim_{h\to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h}
\end{aligned}$$

Since $\lim_{h\to 0} \frac{\sin x(\cos h - 1)}{h} = 0$ and $\lim_{h\to 0} \frac{\cos x \sin h}{h} = \cos x$ both exist, we obtain $\sin' x = \cos x$.

Problem 2. (i) Prove that

$$\cos\frac{x}{2} = \frac{1}{2}\sqrt{2+2\cos x} \quad \text{and} \quad \sin\frac{x}{2} = \frac{1}{2}\sqrt{2-2\cos x}$$

for $0 \le x \le \frac{\pi}{2}$.

(ii) Prove that for every natural number n

$$2^{n-1}\sin\frac{\pi}{2^n}\cos\frac{\pi}{2^2}\cos\frac{\pi}{2^3}\cdots\cos\frac{\pi}{2^n} = 1.$$

Hint: Use (i) and $\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$.

(iii) Use (i) to deduce from (ii) that

$$\frac{2}{\pi}\frac{\pi/2^n}{\sin \pi/2^n} = \frac{\sqrt{2}}{2}\frac{\sqrt{2\sqrt{2}}}{2}\frac{\sqrt{2+\sqrt{2\sqrt{2}}}}{2}\cdots\frac{\sqrt{2\sqrt{2}+\cdots}}{2}$$

where the last factor contains $n-1$ nested square roots.

(iv) Prove Vieta's formula

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2}\frac{\sqrt{2\sqrt{2}}}{2}\frac{\sqrt{2+\sqrt{2\sqrt{2}}}}{2}\cdots$$

Solution.

(i) We have $\cos x = \cos(x/2 + x/2) = 2(\cos x/2)^2 - 1$ from what the stated identity follows. The identity for $\sin x/2$ is proved in a similar way.

(ii) By induction: if $n = 1$,

$$\sin\frac{\pi}{2} = 1$$

Assume that

$$2^{n-2}\sin\frac{\pi}{2^{n-1}}\cos\frac{\pi}{2^2}\cos\frac{\pi}{2^3}\cdots\cos\frac{\pi}{2^{n-1}} = 1$$

and then use (i) to obtain:

$$\begin{aligned}
\sin\frac{\pi}{2^n}\cos\frac{\pi}{2^n} &= \left(\frac{1}{2}\right)\sqrt{2-2\cos\frac{\pi}{2^{n-1}}}\left(\frac{1}{2}\right)\sqrt{2-2\cos\frac{\pi}{2^{n-1}}} \\
&= \frac{1}{2}\sqrt{1-\left(\cos\frac{\pi}{2^{n-1}}\right)^2} \\
&= \frac{1}{2}\sin\frac{\pi}{2^{n-1}}
\end{aligned}$$

The result follows immediately.

(iv) Use (iii) and $\lim_{x\to 0}\frac{x}{\sin x} = 1$.

Problem 3. The objective of this problem is to prove Wallis' Product formula for π.

(i) Prove that

$$\int_0^x \sin^n = -\frac{1}{n}\sin^{n-1} x\cos x + \frac{n-1}{n}\int_0^x \sin^{n-2}.$$

Hint: Use the "integration by parts" technique:

$$\int_a^b uv' = [u(b)v(b) - u(a)v(a)] - \int_a^b u'v.$$

(ii) Let $I_n = \int_0^{\pi/2} \sin^n$. Prove that

$$I_0 = \frac{\pi}{2}, \quad I_1 = 1, \quad \text{and} \quad I_n = \frac{n-1}{n} I_{n-2}.$$

(iii) Prove that

$$\begin{aligned} I_{2n} &= \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{2n-1}{2n}\cdot\frac{\pi}{2} \\ I_{2n+1} &= \frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdots\frac{2n}{2n+1}. \end{aligned}$$

(iv) Prove that

$$0 < I_{2n+2} \le I_{2n+1} \le I_{2n}.$$

Hint: show that

$$0 \le \sin^{2n+2} x \le \sin^{2n+1} x \le \sin^{2n} x$$

for $0 \le x \le \pi/2$.

(v) Prove that

$$\lim_{n\to\infty} \frac{I_{2n}}{I_{2n+1}} = 1.$$

(vi) Prove Wallis' product formula:

$$\frac{\pi}{2} = \lim_{n\to\infty} \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdots\frac{2n}{2n-1}\cdot\frac{2n+1}{2n+1}.$$

Another way of writing Wallis' product formula is

$$\frac{2}{\pi} = \lim_{n\to\infty}\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{4^2}\right)\left(1-\frac{1}{6^2}\right)\cdots\left(1-\frac{1}{(2n)^2}\right).$$

This expression is more interesting because it links Wallis' product formula to Euler's series formula $\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$. (But we have not seen series yet.)

Solution.

(iv) For x in $[0,\pi/2]$ we have $0 \le \sin x \le 1$, hence (multiply this inequality throughout by $\sin x$, and then append it inequality on the right))

$$0 \le (\sin x)^2 \le \sin x \le 1,$$

and so on.

Problem 4. Suppose that f satisfies $f'' = f$ and $f(0) = f'(0) = 0$. Prove that $f(x) = 0$ for all x as follows.

(i) Show that $f^2 = (f')^2$.

(ii) Suppose that $f(x) \neq 0$ for all x in some interval (a,b). Show that there is a constant c such that either $f(x) = ce^x$ for all x in (a,b), or $f(x) = ce^{-x}$ for all x in (a,b).

(iii) Suppose that $f(x_0) \neq 0$ for some x_0. Then $x_0 \neq 0$, say $x_0 > 0$ and thus prove that there is a number a such that $0 \le a < x_0$ and $f(a) = 0$, while $f(x) \neq 0$ for $a < x < x_0$.

(iv) Use (ii) and (iii) to obtain a contradiction (if you assume that $f(x) \neq 0$ for some x.)

Let sinh and cosh be the functions defined by

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad \text{and} \quad \cosh x = \frac{e^x + e^{-x}}{2}$$

called the hyperbolic sine and hyperbolic cosine functions, respectively. There are many analogies between these functions and their trigonometric counterparts sin and cos, You are invited to explore them!

(v) Prove that if f satisfies $f'' = f$, then there are constants a and b such that $f = a\sinh + b\cosh$.

The hyperbolic cosine can be used to study the catenary, or the curve of a hanging chain: what is the shape of the curve assumed by a flexible chain of uniform density which is suspended between two points and hangs under its own weight? If the chain is suspended so that its lowest point is at height $1/a$ at the origin of coordinates, then the shape is that of the graph of the equation $y = \frac{1}{a}\cosh ax$ (if I remember correctly).

Solution.(i) If $g = f^2 - (f')^2$, then $g' = 2ff' - 2f'f'' = 0$. Therefore, g is constant. Since $g(0) = f^2(0) - (f'')^2(0) = 0$, we must have that $g(x) = 0$ for all x, or that $f^2 = (f')^2$.

(ii) Suppose that $f(x) \neq 0$ for all x in (a,b). Then either $f(x) > 0$ for all x in (a,b) or $f(x) < 0$ for all x in (a,b), and similarly for f' because of the identity $f^2 = (f')^2$. Thus either $f = f'$ in (a,b) or $f = -f'$ in (a,b). If $f(x) = f'(x)$ for all x in (a,b), the function g given by $g(x) = f(x)/e^x$ satisfies $g'(x) = 0$ for all x in (a,b), so $g = C$, for some constant $C \neq 0$, or $f(x) = Ce^x$ for all x in (a,b). If $f = -f'$, apply the same argument to the function g given by $g(x) = f(x)/e^{-x}$.

(iii) Suppose that $f(x_0) \neq 0$ for some $x_0 > 0$. Since $f(0) = 0$ and f is continuous, the number $a = \sup\{x \mid x \le x_0 \text{ and } f(x) = 0\}$ satisfies $0 \le a < x_0$, and $f(x) > 0$ for all x in (a, x_0). Apply (ii) to the interval (a, x_0) to obtain that $f(x) = Ce^{\pm x}$ on (a, x_0). We then arrive at a contradiction because, on the one hand, $f(a) = 0$, while on the other hand, $f(a) = \lim_{x\to a} f(x) = Ce^{\pm a} \neq 0$, by continuity.

(iv) Straightforward from (ii) and (iii).

(v) Suppose that $f(0) = b$ and $f'(0) = a$. Then the function $g = f - a\sin h - b\cos h$ satisfies $g'' = g$, $g(0) = 0$ and $g'(0) = 0$. It follows from (i), (ii), (iii), and (iv) that $g = 0$, or that $f = a\sinh + b\cosh$.