Math 512B. Homework 1. Due 1/30/08

(Revised 1/27)

Problem 1. In class we defined $\cos x$ and $\sin x$ for x in $[0, \pi]$. The values of $\cos x$ and $\sin x$ for x not in $[0, \pi]$ are defined in two steps as follows:

(1) If $\pi \leq x \leq 2\pi$, then set

$$\begin{aligned} \cos x &= \cos(2\pi - x), \\ \sin x &= -\sin(2\pi - x) \end{aligned}$$

(2) If $x = 2\pi k + x'$ for some integer k and some x' in $[0, 2\pi]$ then set

$$\begin{aligned} \cos x &= \cos x', \\ \sin x &= \sin x' \end{aligned}$$

Stepts (1) and (2) define cos and sin for all real numbers. We should now check that the basic properties of these functions continue to hold for all real numbers.

(a) Prove that $(\cos x)^2 + (\sin x)^2 = 1$ for all x.

(b) Prove that $cos' x = -\sin x$ and $\sin' x = \cos x$ for all x. (This is easy if x is not a multiple of π.)

Problem 2. If x is not of the form $k\pi + \pi/2$, for any integer k, let

$$\begin{aligned} \tan x &= \frac{\sin x}{\cos x}, \\ \sec x &= \frac{1}{\cos x}. \end{aligned}$$

Prove that

$$\begin{aligned} \tan' x &= (\sec x)^2, \\ \sec' x &= \sec x \tan x \end{aligned}$$

Problem 3. In this problem you will find the derivatives of the inverses of the trigonometric functions cos, sin, and tan (you may also find others by the same method). The trigonometric functions are not one-one, so they must first be restricted to suitable intervals where they are one-one. The largest possible length of such intervals is π and the intervals usually chosen are $[0, \pi]$ for cos, $[-\pi/2, \pi/2]$ for sin, and $(-\pi/2, \pi/2)$ for tan.

The inverse of the function

$$\operatorname{Cos} x = \cos x, \qquad 0 \leq x \leq \pi$$

is denoted by $\arccos x$, and its domain is $[-1, 1]$. The notation $\cos^{-1}$ is avoided because arccos is not the inverse of cos (which does not have an inverse) but of Cos. The inverse of

$$\operatorname{Sin} x = \sin x, \qquad -\pi/2 \leq x \leq \pi/2$$

is denoted by arcsin, and has domain $[-1, 1]$. The inverse of

$$\operatorname{Tan} x = \tan x, \qquad -\pi/2 < x < \pi/2$$

is denoted by arctan, and is defined for all real numbers.

Prove that:

(a) For $-1 < x < 1$, the derivatives

$$\arccos' x = \frac{-1}{\sqrt{1-x^2}}, \quad \text{and} \quad \arcsin' x = \frac{1}{\sqrt{1-x^2}}.$$

(b) For all real numbers x, the derivative

$$\arctan' x = \frac{1}{1+x^2}.$$