Math 512B. Homework 7. Due 4/2/08

Problem 1. For each positive integer N, let

$$F_N(t) = \frac{1}{N+1} \sum_{n=0}^{N} D_n(t)$$

denote the **Fejer kernel**, defined on $[-\pi, \pi]$.

(a) Prove that

$$F_N(t) = \frac{1}{N+1} \frac{\sin^2((N+1)t/2)}{2\sin^2(t/2)}.$$

(b) The function F_N is periodic and non-negative.

(c) The integral $\int_{-\pi}^{\pi} F_N(t)\, dt = 2\pi$.

(d) For each $a > 0$,

$$\lim_{N\to\infty} \int_{a\le|t|\le\pi} F_N(t)\, dt = 0.$$

Problem 2. Let f be integrable on $[-\pi, \pi]$. For each positive integer N, define

$$\sigma_N f(x) = \frac{1}{N+1} \sum_{n=0}^{N} S_n f(x).$$

(a) Prove that

$$\sigma_N f(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} F_N(x-t) f(t)\, dt.$$

(b) Show that if f is continuous on $[-\pi, \pi]$, then $\sigma_N f$ converges to f uniformly on $[-\pi, \pi]$ as $N \to \infty$.

Problem 3. If f has continuous derivative on $[-\pi, \pi]$ then there is a constant M such that $|a_k| + |b_k| \le M/k$ for all $k > 0$.

Problem 4. Let f be a continuous function on $[-\pi, \pi]$, and suppose that all its Fourier coefficients $a_k = 0$ for $k \ge 0$, and $b_k = 0$ for $k \ge 1$. Show that f is identically equal to 0.

Problem 5. Perhaps you have heard about the Dirac delta function. This is the "function" δ on $[-\pi, \pi]$ such that

$$\int_{-\pi}^{\pi} f(x)\delta(x-a)\, dx = f(a).$$

(i) Although δ is not really a function, you may pretend that it is and try to compute its Fourier series. What is this series?

(ii) Suppose that δ was not only just a function, but also a differentaible function with derivative δ'. Compute the Fourier series of δ'.